L^p–REGULARITY FOR PARABOLIC OPERATORS WITH UNBOUNDED TIME–DEPENDENT COEFFICIENTS

MATTHIAS GEISSERT, LUCA LORENZI, AND ROLAND SCHNAUBELT

Abstract. We establish the maximal regularity for nonautonomous Ornstein-Uhlenbeck operators in L^p-spaces with respect to a family of invariant measures, where $p \in (1, +\infty)$. This result follows from the maximal L^p-regularity for a class of elliptic operators with unbounded, time-dependent drift coefficients and potentials acting on $L^p(\mathbb{R}^N)$ with Lebesgue measure.

1. Introduction

In recent years parabolic problems with unbounded time-independent coefficients have been investigated intensively. This line of research has focused on the qualitative behavior, namely on the regularity of solutions and the properties of invariant measures. (See e.g. [4, 6, 12, 39, 42] and the references therein.) Such parabolic problems arise as Kolmogorov equations for ordinary stochastic differential equations. In this context, however, it is natural to consider time-varying coefficients. Recently a corresponding analytical theory for nonautonomous Kolmogorov equations was initiated in [15] (see also [16]). There and in the papers [25, 26] the prototypical case of the nonautonomous Ornstein-Uhlenbeck operator

$$(\mathscr{A}_O(s)\varphi)(x) = \frac{1}{2}\sum_{i,j=1}^N q_{ij}(s) D_{ij}\varphi(x) - \sum_{i,j=1}^N b_{ij}(s) x_j D_i\varphi(x), \qquad x \in \mathbb{R}^N,$$

was studied, assuming that the coefficients q_{ij} and b_{ij} are bounded and continuous in $s \in \mathbb{R}$ and that the matrix $[q_{ij}]$ is symmetric and uniformly positive definite. In this case an explicit formula for the solution of the parabolic equation

$$\begin{cases} D_s u(s,x) = \mathscr{A}_O(s)u(s,x), & s \geq r, \ x \in \mathbb{R}^N, \\ u(r,x) = \varphi(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

when $\varphi \in C_b(\mathbb{R}^N)$ is known. This formula is very useful in many respects (e.g., to study regularity), see [15, 25, 26]; but it will play no role in our investigations. The solutions of (1.1) define evolution operators (or, an evolution family) on $C_b(\mathbb{R}^N)$ by setting $G_O(s,r)\varphi := u(s)$. Recently, the results from [15, 25, 26] have partly been extended to more general elliptic operators with time-varying unbounded coefficients, see [29, 30].

Under suitable assumptions, autonomous Kolmogorov operators admit an invariant measure. As the results in [25] show, this is not true anymore in the nonautonomous case, which is in fact the crucial novelty in the case of time-varying coefficients. However, in [25] it has been proved that it is possible to obtain a *family of invariant measures* $\{\nu_s, \ s \in \mathbb{R}\}$ (also called *evolution system of invariant measures* in [16] and *entrance laws at* $-\infty$ in [23]), provided the matrices $-[b_{ij}(s)]$

Key words and phrases. Nonautonomous Ornstein-Uhlenbeck operators, Kolmogorov equations, invariant measures, evolution operators, evolution semigroups.

L.L. wishes to thank the Department of Mathematics of the University of Karlsruhe, where this paper originated, for the kind hospitality during his visit.

generate an exponentially stable evolution family on $\mathbb{R}^N$. These measures are Borel probability measures on $\mathbb{R}^N$ satisfying the equation

$$\int_{\mathbb{R}^N} G_O(s,r)\varphi\, d\nu_s = \int_{\mathbb{R}^N} \varphi\, d\nu_r \tag{1.2}$$

for all $\varphi \in C_b(\mathbb{R}^N)$ and all $r, s \in \mathbb{R}$ with $r \leq s$. The set of all such families of invariant measures has been described in [25], and it was shown that there exists exactly one family $\{\mu_s,\ s \in \mathbb{R}\}$ of Gaussian type which has finite moments of every order. In formula (2.1) we recall the explicit formula for μ_s. The existence of families of invariant measures for more general nonautonomous operators has recently been proved in [29], see also [5, 7, 8] for related results.

The defining property (1.2) of invariant measures easily implies that one can extend the evolution operator associated with (1.1) to a contraction $G_O(s,r) : L^p(\mathbb{R}^N, \mu_r) \to L^p(\mathbb{R}^N, \mu_s)$ for all $s \geq r$. As in the autonomous case one can expect good regularity properties of this extension. But in the nonautonomous case one has to pay the price that the evolution operators act on a family of spaces. In addition, it is well known that the asymptotic behavior of nonautonomous problems is much more difficult to treat than in the autonomous case. For an evolution family on a fixed Banach space an associated 'evolution semigroup' was introduced for the study of evolution families. For instance, this semigroup allows to derive spectral theoretic characterizations of certain asymptotic properties of the evolution family, see [13], [40]. It was observed by Da Prato and Lunardi in [15] that one can generalize this construction also to the case of L^p-spaces with time-varying measures, and the authors used the evolution semigroup in the study of longterm behavior of G_O, see also [25, 26, 29].

Following these papers, we define a measure ν on Borel sets on $\mathbb{R}^{1+N}$ by setting

$$\nu(J \times B) = \int_J \mu_s(B)\, ds \tag{1.3}$$

for Borel sets $B \subset \mathbb{R}^N$ and $J \subset \mathbb{R}$. Of course, ν is not a probability measure anymore. One further introduces the *evolution semigroup* $T(\cdot)$ on $L^p(\mathbb{R}^{1+N}, \nu)$ corresponding to G_O defined by

$$(T(t)f)(s,x) = (G_O(s, s-t)f(s-t, \cdot))(x), \qquad (s,x) \in \mathbb{R}^{1+N},\ t \geq 0, \tag{1.4}$$

where $f \in L^p(\mathbb{R}^{1+N}, \nu)$ and $1 \leq p < +\infty$. It is straightforward to check that equation (1.4) defines in fact a C_0-semigroup on $L^p(\mathbb{R}^{1+N}, \nu)$ and that

$$\int_{\mathbb{R}^{1+N}} T(t)f\, d\nu = \int_{\mathbb{R}^{1+N}} f\, d\nu, \qquad t > 0,\ f \in C_c(\mathbb{R}^{1+N}),$$

see [15] or [25]. We denote the generator of $T(\cdot)$ in $L^p(\mathbb{R}^{1+N}, \nu)$ by G_p, where $1 \leq p < +\infty$. In [30] it has been proved that G_p is the closure the parabolic operator $\mathscr{G}$ defined by

$$(\mathscr{G}u)(s,x) = (\mathscr{A}_O(s)u(s,\cdot))(x) - D_s u(s,x), \qquad (s,x) \in \mathbb{R}^{1+N},$$

for $u \in C_c^\infty(\mathbb{R}^{1+N})$. In this paper we want to show that G_p has the 'natural' domain

$$\begin{aligned} D(G_p) &= \{u \in L^p(\mathbb{R}^{1+N}, \nu) : D_t, D_i u, D_{ij} u \in L^p(\mathbb{R}^{1+N}, \nu),\ \forall\, i,j = 1, \dots, N\} \\ &=: W_p^{1,2}(\mathbb{R}^{1+N}, \nu), \end{aligned} \tag{1.5}$$

for $1 < p < +\infty$, see Theorem 3.11. This means that for each inhomogeneity $f \in L^p(\mathbb{R}^{1+N}, \nu)$ and each $\lambda > 0$, the function $u = (\lambda - G_p)^{-1} f$ is the only solution in $W_p^{1,2}(\mathbb{R}^{1+N}, \nu)$ of the parabolic equation

$$D_s u(s) = (\mathscr{A}_O(s) - \lambda)u(s) + f(s), \qquad s \in \mathbb{R}, \tag{1.6}$$

on the line. In other words, the problem (1.6) possesses maximal L^p-regularity with respect to the measure ν. Such results are known in the autonomous case even in much greater generality, see e.g. [14, 17, 32, 33, 34, 35] and the references therein, where a variety of methods was developed. In the case $p = 2$, the identity (1.5) was shown in [25] for the nonautonomous case using regularity properties of $G_O(s, r)$ and tools from interpolation theory. However, the necessary results from interpolation theory do not hold for $p \neq 2$.

In this paper we establish (1.5) for all $p \in (1, +\infty)$ using a completely different method, inspired by [17] and [35]. We transform the operator $\mathscr{G}$ into an operator $\mathscr{L}_O$ on the space $L^p(\mathbb{R}^{1+N})$ with Lebesgue measure which has a dominating potential, see Section 2. The operator $\mathscr{L}_O$ is a (simple) special case of a class of parabolic operators $\mathscr{L} = \mathscr{A}(\cdot) - D_s$ on $L^p(\mathbb{R}^{1+N})$ with time-varying coefficients, see (3.1). The uniformly elliptic operators $\mathscr{A}(s)$ may have unbounded potential and drift coefficients. We require that the potential satisfies an oscillation condition and that it dominates the drift coefficients, as described in Section 3. In Theorem 3.8 it is shown that the realization L_p of $\mathscr{L}$ in $L^p(\mathbb{R}^{1+N})$, where $1 < p < +\infty$, with the domain $D(L_p) = W_p^{1,2}(\mathbb{R}^{1+N}) \cap D(V) =: \mathscr{D}_p$ generates a positive and contractive evolution semigroup $S(\cdot)$. Hence the parabolic equation

$$D_s u(s) = (\mathscr{A}(s) - \lambda)u(s) + f(s), \qquad s \in \mathbb{R}, \tag{1.7}$$

has the unique solution $u = (\lambda - L_p)^{-1} f$ in $\mathscr{D}_p$, for every $f \in L^p(\mathbb{R}^{1+N})$ and $\lambda > 0$; i.e., (1.7) has maximal L^p-regularity. Moreover, the evolution family associated with $S(\cdot)$ solves the initial value problem corresponding to (1.7). In Section 4 we extend these results to the spaces $L^1(\mathbb{R}^{1+N})$ and $C_0(\mathbb{R}^{1+N})$.

By means of Theorem 3.8 one could also treat generalized Ornstein-Uhlenbeck operators as in [17, 35]. For simplicity, we restrict ourselves to the basic and most prominent case of the classical Ornstein-Uhlenbeck operators.

Our main theorems are based on two crucial estimates and on semigroup theory. In Proposition 3.4 we show a weighted gradient estimate which allows to control the gradient term by the heat operator and the potential. Proposition 3.7 gives the main *a priori* estimate for the parabolic operator $\mathscr{L}$ which implies that its realization L_p with domain $\mathscr{D}_p$ is closed in $L^p(\mathbb{R}^{1+N})$. We then verify that L_p is maximally dissipative and employ the theory of evolution semigroups to establish Theorem 3.8. The proofs for the spaces L^1 and C_0 in the fourth section are similar, and the one for C_0 uses the L^p result. Our approach is inspired by the paper [35] which was devoted to the autonomous case, but there are fundamental differences. So we cannot use the theory of analytic semigroups since the evolution semigroup $S(\cdot)$ is not analytic. (The spectrum of its generator contains vertical lines, see [13, Theorem 3.13].) Further, the known results on parabolic evolution operators do not apply to the class of elliptic operators $\mathscr{A}(s)$ studied here, see Remark 3.9. Moreover, the presence of the time derivative in $\mathscr{L}$ leads to new difficulties in the proofs of Propositions 3.4 and 3.7. For instance, we need a parabolic version of the Besicovitch covering theorem established in the Appendix.

Besides [35] and the papers mentioned above, there are several works treating L^p-regularity for autonomous problems with unbounded coefficients in L^p-spaces with respect to the Lebesgue measure, see e.g. [9, 10, 24, 37] and the references therein. We are only aware of one related paper for nonautonomous problems (except for [26]): in [11] operators without drift terms were studied with completely different methods and assumptions.

Notations. We denote by $|\cdot|$ the Euclidean norm of vectors, whereas $\|A\|$ is the operator norm of a matrix with respect to the Euclidean norm. The transpose of A is A^* and $\mathrm{Tr}A$ is its trace. Open balls in $\mathbb{R}^d$ are designated by $B(x, r)$. We write

$\langle \xi, \eta \rangle$ or $\xi \cdot \eta$ for the scalar product in $\mathbb{R}^d$ and I for the identity map. D_j, ∇, D^2 and div are the (distributional) partial derivatives, gradient, Hessian matrix and divergence, respectively, with respect to the space variable $x \in \mathbb{R}^N$. We also use the notations ∇_x, D_x^2 and div_x if a function depends on both the time and space variables $(s,x) \in \mathbb{R}^{1+N}$. In this case D_s is the time derivative. We always denote the spatial Laplace operator by $\Delta = D_1^2 + \cdots + D_N^2$.

In this paper we only consider real function spaces. The symbol C^k refers to spaces of k-times continuously differentiable functions, where $k \in \mathbb{N} \cup \{0, +\infty\}$. In such spaces the subscript c means 'with compact support', whereas the subscript b (resp. 0) means that the functions and the derivatives up to order k are bounded (resp. vanish at ∞). The space of continuous functions $f : \mathbb{R}^{1+N} \to \mathbb{R}^d$ such that also $\nabla_x f$ is continuous on $\mathbb{R}^{1+N}$ is denoted by $C^{0,1}(\mathbb{R}^{1+N}, \mathbb{R}^d)$. Let μ be a σ-finite measure on $\mathbb{R}^{1+N}$. Then, $W_p^{1,2}(\mathbb{R}^{1+N}, \mu)$ is the space of functions $f : \mathbb{R}^{1+N} \to \mathbb{R}$ such that f and the distributional derivatives $D_s f$, $D_j f$ and $D_{ij} f$ $(i,j = 1, \dots, N)$ belong to $L^p(\mathbb{R}^{1+N}, \mu)$. We endow it with the natural norm

$$\begin{aligned}\|f\|^p_{W_p^{1,2}(\mathbb{R}^{1+N},\mu)} &= \int_{\mathbb{R}^{1+N}} |f|^p d\mu + \int_{\mathbb{R}^{1+N}} |D_s f|^p d\mu \\ &\quad + \sum_{j=1}^N \int_{\mathbb{R}^{1+N}} |D_j f|^p d\mu + \sum_{i,j=1}^N \int_{\mathbb{R}^{1+N}} |D_{ij} f|^p d\mu.\end{aligned}$$

We use analogous definitions for subsets of the form $(a,b) \times \mathbb{R}^N$. If μ is the Lebesgue measure, we omit μ in the notation. The usual isotropic Sobolev spaces on $\mathbb{R}^d$ are denoted by $W_p^k(\mathbb{R}^d)$. The norm on $L^p(\mathbb{R}^d)$ is designated by $\|f\|_p$ for $1 \le p \le +\infty$. Finally, we write $c = c(\alpha, \dots)$ for a constant depending only on the quantities $\alpha, \dots$ Such constants may vary from line to line.

2. Transformation of the parabolic Ornstein–Uhlenbeck operator

For any continuous function $s \mapsto B(s)$ from $\mathbb{R}$ into the set of $N \times N$ matrices, we denote by $U(s,r)$ the solution of the problem

$$\begin{cases} D_s U(s,r) = B(s) U(s,r), \quad s \in \mathbb{R}, \\ U(r,r) = I, \end{cases}$$

where $r \in \mathbb{R}$. We state our hypotheses on the coefficients $Q(s) = [q_{ij}(s)]$ and $B(s) = [b_{ij}(s)]$ of the Ornstein-Uhlenbeck operator $\mathscr{A}_O(s)$.

Hypothesis 2.1. (i) The coefficients q_{ij} and b_i belong, respectively, to $C_b^1(\mathbb{R})$ and $C_b(\mathbb{R})$ for all $i,j = 1, \dots, N$.

(ii) For every $s \in \mathbb{R}$, the matrix $Q(s)$ is symmetric and there exists a constant $\eta_0 > 0$ such that

$$\langle Q(s)\xi, \xi \rangle \ge \eta_0 |\xi|^2, \qquad \xi \in \mathbb{R}^N, \ s \in \mathbb{R}.$$

(iii) There exist constants $C_0, \omega > 0$ such that

$$\|U(r,s)\| \le C_0 e^{-\omega(s-r)}, \qquad s, r \in \mathbb{R} \text{ with } s \ge r.$$

Under the above assumptions, there exists a family of invariant measures for $\mathscr{A}_O(s)$ (see (1.2)) of Gaussian type given by

$$\mu_s(dx) = (2\pi)^{-\frac{N}{2}} (\det Q_s)^{-\frac{1}{2}} e^{-\frac{1}{2}\langle Q_s^{-1} x, x \rangle}, \qquad s \in \mathbb{R}, \ x \in \mathbb{R}^N, \tag{2.1}$$

$$Q_s = \int_s^{+\infty} U(s,\xi) Q(\xi) U^*(s,\xi) d\xi, \qquad s \in \mathbb{R}, \tag{2.2}$$

see [15] and [25]. Actually, the authors of the previous papers deal with backward nonautonomous parabolic problems, whereas we have preferred to consider forward

problems in the present paper. But a straightforward change of variables allows to transform the problem (1.1) into a backward Cauchy problem. More precisely, for any $r \in \mathbb{R}$, the function $(s,x) \mapsto v(s,x) := (G_O(-s,-r)\varphi)(x)$ is a classical solution to the backward Cauchy problem

$$\begin{cases} D_s v(s,x) + \hat{\mathscr{A}}_O(s)v(s,x) = 0, & s \le r,\ x \in \mathbb{R}^N, \\ v(r,x) = \varphi(x), & x \in \mathbb{R}^N, \end{cases} \tag{2.3}$$

where

$$\hat{\mathscr{A}}_O(s)\varphi \;=\; \frac{1}{2}\sum_{i,j=1}^N q_{ij}(-s)D_{ij}\varphi - \sum_{i=1}^N b_{ij}(-s)x_j D_i\varphi.$$

Hence, the evolution operator $G_O(s,r)$ associated with problem (1.1) and the evolution operator $P(s,r)$ associated with problem (2.3) are related by the formula

$$G_O(s,r)\varphi = P(-s,-r)\varphi, \qquad r \le s,\ \varphi \in C_b(\mathbb{R}^N).$$

In the first lemma we collect some estimates concerning the densities of the invariant measures.

Lemma 2.2. *Assume that Hypothesis 2.1 is satisfied. Then, there exist two constants $C_1, C_2 > 0$ such that the inequalities*

$$C_1|x|^2 \le \langle Q_r x, x\rangle \le C_2|x|^2, \tag{2.4}$$

$$C_2^{-1}\,|x| \le |Q_r^{-1}x| \le C_1^{-1}\,|x|, \tag{2.5}$$

$$C_1^N \le \det Q_r \le C_2^N, \tag{2.6}$$

hold for all $r \in \mathbb{R}$ and $x \in \mathbb{R}^N$.

Proof. Let $x \in \mathbb{R}^N$ and $r \in \mathbb{R}$. Formula (2.2) and Hypothesis 2.1 yield that

$$\begin{aligned} \langle Q_r x, x\rangle &= \int_r^{+\infty} \langle Q(\xi)U^*(r,\xi)x, U(r,\xi)^*x\rangle\, d\xi \\ &\le C_0^2\|Q\|_\infty |x|^2 \int_r^{+\infty} e^{-2\omega(\xi-r)}d\xi = \frac{C_0^2}{2\omega}\|Q\|_\infty\,|x|^2, \end{aligned} \tag{2.7}$$

for any $x \in \mathbb{R}^N$, which accomplishes the proof of the second inequality in (2.4) with $C_2 = \frac{C_0^2}{2\omega}\|Q\|_\infty$. We further recall that $U(r,s)^{-1} = U(s,r)$ for all $r,s \in \mathbb{R}$ and that $\|U(r,s)\| \le M_0 e^{\varpi(r-s)}$ for constants $\varpi \in \mathbb{R}_+$ and $M_0 \ge 1$ and all $r \ge s$. It thus holds

$$|x| = |U^*(\xi,r)U^*(r,\xi)x| \le \|U(\xi,r)\|\;|U^*(r,\xi)x| \le M_0 e^{\varpi(\xi-r)}\,|U^*(r,\xi)x|,$$

for all $r,\xi \in \mathbb{R}$ with $r \le \xi$ and all $x \in \mathbb{R}^N$. Using (2.7) and Hypothesis 2.1(ii), we then deduce

$$\langle Q_r x, x\rangle \ge \eta_0 \int_r^{+\infty} |U^*(r,\xi)x|^2\, d\xi \ge \frac{\eta_0|x|^2}{M_0^2}\int_r^{+\infty} e^{-2\varpi(\xi-r)}\, d\xi = \frac{\eta_0}{2M_0^2\varpi}|x|^2,$$

which gives the first estimate in (2.4) with $C_1 = \eta_0(2M_0^2\varpi)^{-1}$. The assertion (2.4) is equivalent to

$$\sqrt{C_1}\,|x| \le |Q_r^{1/2}x| \le \sqrt{C_2}\,|x|. \tag{2.8}$$

The first inequality in (2.5) now follows noting that

$$|x| = |Q_r^{1/2}Q_r^{1/2}Q_r^{-1}x| \le \|Q_r^{1/2}\|^2\;|Q_r^{-1}x| \le C_2\,|Q_r^{-1}x|.$$

On the other hand, (2.8) implies $\|Q_r^{-1/2}\| \le C_1^{-1/2}$ and, hence, the second part of (2.5). The final assertion (2.6) is a consequence of the fact that the eigenvalues of Q_r belong to the interval $[C_1, C_2]$ due to (2.4). $\square$

Let $p \in (1, +\infty)$. We now transform the differential operator $\mathscr{G} = \mathscr{A}_O(\cdot) - D_s$ acting on $L^p(\mathbb{R}^{1+N}, \nu)$ into a differential operator $\mathscr{L}_O$ acting on $L^p(\mathbb{R}^{1+N})$. To this purpose, we set $\Phi(s,x) = \frac{1}{2}\langle Q_s^{-1}x, x\rangle$ for $(s,x) \in \mathbb{R}^{1+N}$. Observe that (1.3), (2.1) and (2.6) yield

$$\int_{\mathbb{R}^{1+N}} |e^{\frac{1}{p}\Phi} f|^p \, d\nu = (2\pi)^{-\frac{N}{2}} \int_{\mathbb{R}^{1+N}} (\det Q_s)^{-\frac{1}{2}} |f|^p \, ds \, dx \le (2\pi C_1)^{-\frac{N}{2}} \int_{\mathbb{R}^{1+N}} |f|^p ds \, dx,$$

$$\int_{\mathbb{R}^{1+N}} |e^{-\frac{1}{p}\Phi} g|^p \, ds \, dx = (2\pi)^{\frac{N}{2}} \int_{\mathbb{R}^{1+N}} (\det Q_s)^{\frac{1}{2}} |g|^p \, d\nu \le (2\pi C_2)^{\frac{N}{2}} \int_{\mathbb{R}^{1+N}} |g|^p \, d\nu,$$

for every $f \in L^p(\mathbb{R}^{1+N})$ and $g \in L^p(\mathbb{R}^{1+N}, \nu)$. Therefore the operator $M_p : L^p(\mathbb{R}^{1+N}) \to L^p(\mathbb{R}^{1+N}, \nu)$, defined by

$$(M_p f)(s,x) = e^{\frac{1}{2p}\langle Q_s^{-1}x, x\rangle} f(s,x) = e^{\frac{1}{p}\Phi(s,x)} f(s,x), \tag{2.9}$$

is an isomorphism with the inverse $M_p^{-1} g = e^{-\frac{1}{p}\Phi} g$. On test functions we now define the differential operator

$$\mathscr{L}_O := M_p^{-1}(\mathscr{A}_O(\cdot) - D_s) M_p. \tag{2.10}$$

Let u be smooth. A straightforward computation shows that the equalities

$$\begin{aligned}
D_s M_p u &= \frac{1}{p}(M_p u) D_s \Phi + M_p(D_s u),\\
D_i M_p u &= \frac{1}{p} e^{\frac{1}{p}\Phi} u D_i \Phi + e^{\frac{1}{p}\Phi} D_i u = \frac{1}{p}(M_p u) D_i \Phi + M_p(D_i u),\\
D_{ij} M_p u &= \frac{1}{p}(M_p u) D_{ij}\Phi + \frac{1}{p^2}(M_p u)(D_i\Phi) D_j \Phi + \frac{1}{p}(D_i\Phi) M_p(D_j u)\\
&\qquad + \frac{1}{p}(D_j \Phi) M_p(D_i u) + M_p(D_{ij} u)
\end{aligned} \tag{2.11}$$

hold on $\mathbb{R}^{1+N}$ for all $i,j = 1,\dots,N$. For $(s,x) \in \mathbb{R}^{1+N}$ we thus obtain

$$\begin{aligned}
(\mathscr{L}_O u)(s,x) &= -D_s u(s,x) + \frac{1}{2}\mathrm{Tr}(Q(s) D_x^2 u(s,x)) - \langle B(s)x, \nabla_x u(s,x)\rangle\\
&\qquad + \frac{1}{p}\langle Q(s)\nabla_x \Phi(s,x), \nabla_x u(s,x)\rangle + \frac{1}{2p}\mathrm{Tr}(Q(s) D_x^2 \Phi(s,x)) u(s,x)\\
&\qquad + \frac{1}{2p^2}\langle Q(s)\nabla_x\Phi(s,x), \nabla_x \Phi(s,x)\rangle u(s,x) - \frac{1}{p} u(s,x) D_s \Phi(s,x)\\
&\qquad - \frac{1}{p}\langle B(s)x, \nabla_x \Phi(s,x)\rangle u(s,x)\\
&=: -D_s u(s,x) + \frac{1}{2}\mathrm{Tr}\left(Q(s) D_x^2 u(x)\right) + \langle F_O(s,x), \nabla_x u(s,x)\rangle - V_O(s,x) u(s,x).
\end{aligned}$$

To write F_O and V_O more conveniently, we observe that

$$\nabla_x \Phi(s,x) = Q_s^{-1} x \qquad \text{and} \qquad D_x^2 \Phi(s,x) = Q_s^{-1}. \tag{2.12}$$

As a consequence,

$$F_O(s,x) = \frac{1}{p} Q(s) Q_s^{-1} x - B(s)x. \tag{2.13}$$

We further have

$$\begin{aligned}
D_s \Phi(s,x) &= \frac{1}{2}\langle D_s Q_s^{-1} x, x\rangle = -\frac{1}{2}\langle Q_s^{-1}\left(D_s Q_s\right) Q_s^{-1} x, x\rangle\\
&= -\frac{1}{2}\left\langle Q_s^{-1}\left(D_s \int_s^{+\infty} U(s,r) Q(r) U^*(s,r)\, dr\right) Q_s^{-1} x, x\right\rangle\\
&= -\frac{1}{2}\langle Q_s^{-1}\left(-Q(s) + B(s) Q_s + Q_s B^*(s)\right) Q_s^{-1} x, x\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2}\big(\langle Q(s)Q_s^{-1}x, Q_s^{-1}x\rangle - \langle Q_s^{-1}B(s)x, x\rangle - \langle B^*(s)Q_s^{-1}x, x\rangle\big)\\
&= \frac{1}{2}\langle Q(s)Q_s^{-1}x, Q_s^{-1}x\rangle - \langle Q_s^{-1}B(s)x, x\rangle. \qquad (2.14)
\end{aligned}
$$

It follows that

$$
\begin{aligned}
V_O(s,x) &= -\frac{1}{2p}\operatorname{Tr}(Q(s)Q_s^{-1}) - \frac{1}{2p^2}\langle Q(s)Q_s^{-1}x, Q_s^{-1}x\rangle + \frac{1}{p}\langle B(s)x, Q_s^{-1}x\rangle\\
&\qquad + \frac{1}{p}\left(\frac{1}{2}\langle Q(s)Q_s^{-1}x, Q_s^{-1}x\rangle - \langle Q_s^{-1}B(s)x, x\rangle\right)\\
&= \frac{1}{2p}\left(1-\frac{1}{p}\right)\langle Q(s)Q_s^{-1}x, Q_s^{-1}x\rangle - \frac{1}{2p}\operatorname{Tr}(Q(s)Q_s^{-1}), \qquad (2.15)
\end{aligned}
$$

for all $(s,x) \in \mathbb{R}^{1+N}$. Now, let $p \in (1,+\infty)$. Hypothesis 2.1 and Lemma 2.2 then imply that $V_O(s,x) \ge k_1\,|x|^2 - k_0$ for constants $k_1 = k_1(p) > 0$ and $k_0 \ge 0$ and all $(s,x) \in \mathbb{R}^{1+N}$. We fix the number $c_0 = 2\|\operatorname{div}_x F_O\|_\infty$ (which is possible because of Hypothesis 2.1, Lemma 2.2 and (2.13)) and set

$$
W_O(s,x) = c_0 + k_1\,|x|^2,
$$

for all $(s,x) \in \mathbb{R}^{1+N}$. In view of Lemma 2.2 and formulas (2.13) and (2.15), there exist constants $\lambda = k_0 + c_0 \ge 0$, $c_1 = c_1(p) \ge 1$, $\kappa = \kappa(p) > 0$, and $\theta = 2/3$ with

$$
W_O \le \lambda + V_O \le c_1 W_O, \qquad |F_O| \le \kappa W_O^{1/2}, \qquad \theta W_O + \operatorname{div}_x F_O \ge 0 \qquad (2.16)
$$

on $\mathbb{R}^{1+N}$.

3. Operators with dominating potential for $1 < p < +\infty$.

In this section we mainly consider elliptic operators of the form

$$
\mathscr{A}(s)\varphi = \operatorname{div}_x(a(s)\nabla_x\varphi) + F(s)\cdot\nabla_x\varphi - V(s)\varphi, \qquad (3.1)
$$

at first defined for $\varphi \in C_c^\infty(\mathbb{R}^N)$, and their parabolic counterpart

$$
\mathscr{L}u = (\mathscr{A}(\cdot) - D_s)u,
$$

at first defined for $u \in C_c^\infty(\mathbb{R}^{1+N})$. We assume the following conditions on the coefficients $a = [a_{ij}]$, F and V.

- **(A1)** $a_{ij} \in C_b^1(\mathbb{R}^{1+N})$ satisfy $a_{ij} = a_{ji}$ and
$$
\sum_{i,j=1}^N a_{ij}(s,x)\xi_i\xi_j \ge \eta_0|\xi|^2,
$$
for all $x, \xi \in \mathbb{R}^N$, $s \in \mathbb{R}$, $i,j \in \{1,\dots,N\}$ and some constant $\eta_0 > 0$.
- **(A2)** $W \in C^1(\mathbb{R}^{1+N})$ is a function such that $W \ge c_0 > 0$, $|D_sW| \le \beta W^2 + K_\beta$ and $|\nabla_x W| \le \gamma W^{\frac{3}{2}} + K'_\gamma$ for some constants $c_0, \beta, \gamma > 0$ and $K_\beta, K'_\gamma \ge 0$.
- **(A3)** $V : \mathbb{R}^{1+N} \to \mathbb{R}$ is measurable and $W \le V \le c_1 W$ for some constant $c_1 \ge 1$.
- **(A4)** $F \in C(\mathbb{R}^{1+N}, \mathbb{R}^N)$ satisfies $|F| \le \kappa W^{\frac{1}{2}}$ for some constant $\kappa \ge 0$.
- **(A5)** $F \in C^{0,1}(\mathbb{R}^{1+N}, \mathbb{R}^N)$ and there exists a constant $\theta \in [0,p)$ such that $\theta W + \operatorname{div}_x F \ge 0$, where $p \in [1,+\infty)$ is given.

Later on we will impose additional restrictions on the size of β and γ, see (3.7). Due to (2.16) the functions Q, F_O, $\lambda + V_O$ and W_O from the previous section satisfy (A1)–(A5) with $\beta = K_\beta = 0$ and arbitrarily small $\gamma > 0$, for each $p \in (1,+\infty)$. Except for the estimate on D_sW, the hypotheses (A1)–(A5) were already used in [35] in the autonomous case. We want to discuss them shortly, referring the reader to [35] for more details and further references. Of course, (A1) gives uniform ellipticity. Assumption (A4) allows us to control the drift term by the potential. (But note that the drift term is not a small perturbation, cf. [35, Remark 3.6].)

The inequality in (A5) is a slightly strengthened dissipativity condition for $\mathscr{L}$. The crucial hypothesis is (A2) which restricts the oscillation of the auxiliary potential W, whereas (A3) allows to compare V and W. The use of W gives some more flexibility in the applications (as already exploited in [35, Section 7]). Example 3.7 in [35] shows that one cannot omit (A2) and that even the restriction in (3.7) is almost sharp in certain cases.

In this section we want to show that $\mathscr{L}$, endowed with the domain

$$\mathscr{D}_p := \{u \in W^{1,2}_p(\mathbb{R}^{1+N}) : Wu \in L^p(\mathbb{R}^{1+N})\},$$

generates a C_0-semigroup on $L^p(\mathbb{R}^{1+N})$ and we want to exploit this fact in the study of (1.6) and its variant (3.13) for $\mathscr{A}(\cdot)$. In the next section we also use the domains

$$\begin{aligned}
\mathscr{D}_1 &:= \{u \in L^1(\mathbb{R}^{1+N}) : (\Delta - D_s)u, Wu \in L^1(\mathbb{R}^{1+N})\},\\
\mathscr{D}_\infty &:= \{u \in C_0(\mathbb{R}^{1+N}) : (\Delta - D_s)u, Wu \in C_0(\mathbb{R}^{1+N})\}\\
&= \{u \in C_0(\mathbb{R}^{1+N}) \cap W^{1,2}_{q,\mathrm{loc}}(\mathbb{R}^{1+N})\ \forall\, q < +\infty : (\Delta - D_s)u, Wu \in C_0(\mathbb{R}^{1+N})\},
\end{aligned}$$

where the last equality follows from standard local parabolic regularity. The spaces $\mathscr{D}_p$, $1 \le p \le +\infty$, are endowed with their natural norms given by

$$\begin{aligned}
\|u\|^p_{\mathscr{D}_p} &= \|u\|^p_{W^{1,2}_p(\mathbb{R}^{1+N})} + \|Wu\|^p_p, \qquad 1 < p < +\infty,\\
\|u\|_1 &= \|(\Delta - D_s)u\|_1 + \|Wu\|_1,\\
\|u\|_\infty &= \max\{\|(\Delta - D_s)u\|_\infty\,, \|Wu\|_\infty\}.
\end{aligned}$$

Note that in the definitions of the spaces $\mathscr{D}_p$ and their norms, one could replace everywhere W by V getting the same sets and equivalent norms (where V is assumed to be continuous if $p = \infty$). We recall that the norm on $W^{1,2}_p(\mathbb{R}^{1+N})$ is equivalent to the graph norm of $\Delta - D_s$ on $L^p(\mathbb{R}^{1+N})$ if $p \in (1, +\infty)$. At first, we prove three more or less standard facts for every $p \in [1, +\infty]$,

Lemma 3.1. *Assume that hypothesis* (A1) *is satisfied. Then, the following assertions hold.*

(a) If $F \in C^{0,1}(\mathbb{R}^{1+N}, \mathbb{R}^N)$, $V \in L^p_{\mathrm{loc}}(\mathbb{R}^{1+N})$, and $V + \frac{1}{p}\operatorname{div}_x F \ge 0$ for some $1 \le p < +\infty$, then $(\mathscr{L}, C^\infty_c(\mathbb{R}^{1+N}))$ is dissipative in $L^p(\mathbb{R}^{1+N})$.

(b) If $F \in C(\mathbb{R}^{1+N}, \mathbb{R}^N)$, $V \in C(\mathbb{R}^{1+N})$ and $V \ge 0$, then $(\mathscr{L}, C^\infty_c(\mathbb{R}^{1+N}))$ is dissipative in $C_0(\mathbb{R}^{1+N})$.

Proof. Let $1 \le p < +\infty$. It is known that

$$\int_{\mathbb{R}^N} (\mathscr{A}(s)\varphi)\, \varphi|\varphi|^{p-2}\, dx \le 0,$$

for all $\varphi \in C^\infty_c(\mathbb{R}^N)$ and $s \in \mathbb{R}$, see e.g., [35, Lemma 2.6]. For $u \in C^\infty_c(\mathbb{R}^{1+N})$ we thus obtain

$$\int_{\mathbb{R}^{1+N}} (\mathscr{A}(\cdot)u - D_s u)\, u|u|^{p-2}\, ds\, dx \le -\frac{1}{p}\int_{\mathbb{R}^{1+N}} D_s\, |u|^p\, ds\, dx = 0.$$

This shows assertion (a). The dissipativity of $\mathscr{L}$ in $C_0(\mathbb{R}^{1+N})$ is a standard consequence of the maximum principle. □

Lemma 3.2. *For every $u \in C^\infty_c(\mathbb{R}^{1+N})$ and $1 \le p \le +\infty$, we have*

$$\|\nabla_x u\|_p \le C\, \|(\Delta - D_s)u\|_p^{\frac{1}{2}}\, \|u\|_p^{\frac{1}{2}},$$

with a constant $C > 0$ depending only on N.

Proof. For a given $\lambda > 0$ and $u \in C_c^\infty(\mathbb{R}^{1+N})$, we set $f = \lambda u - (\Delta - D_s)u$. Let $G_p(\cdot)$ be the heat semigroup generated by Δ on $L^p(\mathbb{R}^N)$ for $1 \le p < +\infty$, and on $C_0(\mathbb{R}^N)$ for $p = +\infty$, respectively. The variation of constants formula yields

$$u(t) = \int_{-\infty}^t e^{-\lambda(t-s)} G_p(t-s) f(s)\, ds,$$

for all $t \in \mathbb{R}$. Using the well known estimate $\sqrt{s}\, \|\nabla_x G_p(s)\varphi\|_p \le c\, \|\varphi\|_p$ valid for every $s > 0$ and $\varphi \in L^p(\mathbb{R}^N)$ or $\varphi \in C_0(\mathbb{R}^N)$, respectively (where $c = c(N)$ is a constant), we deduce that

$$\|\nabla_x u(t)\|_p \le \int_{-\infty}^t \frac{ce^{-\lambda(t-s)}}{\sqrt{t-s}} \|f(s)\|_p\, ds,$$

for all $t \in \mathbb{R}$. Young's inequality then implies

$$\|\nabla_x u\|_p \le \frac{c\sqrt{\pi}}{\sqrt{\lambda}}\, \|f\|_p \le \frac{c\sqrt{\pi}}{\sqrt{\lambda}}\, (\lambda\, \|u\|_p + \|(\Delta - D_s)u\|_p),$$

for each $\lambda > 0$. The assertion follows if we take $\lambda = \|(\Delta - D_s)u\|_p\, \|u\|_p^{-1}$. □

Lemma 3.3. *Assume that $W \in C(\mathbb{R}^{1+N})$ satisfies $W \ge c_0 > 0$. Then, $C_c^\infty(\mathbb{R}^{1+N})$ is dense in $\mathscr{D}_p$ for $1 \le p \le +\infty$.*

Proof. Let η be a cutoff function on $\mathbb{R}^{1+N}$ such that $\mathbb{1}_{B(0,1)} \le \eta \le \mathbb{1}_{B(0,2)}$. Define $\eta_n(t,x) = \eta(t/n, x/n)$ for all $(t,x) \in \mathbb{R}^{1+N}$ and $n \in \mathbb{N}$. Let $u \in \mathscr{D}_p$. Then, $\eta_n u \to u$ and $W\eta_n u \to Wu$ as $n \to +\infty$ in $L^p(\mathbb{R}^{1+N})$. Moreover,

$$(D_s - \Delta)(\eta_n u) = \eta_n(D_s - \Delta)u + u(D_s - \Delta)\eta_n - 2\langle \nabla_x u, \nabla_x \eta_n\rangle.$$

Since the derivatives of η_n tend to 0 in the sup-norm as $n \to +\infty$, the functions $(D_s - \Delta)(\eta_n u)$ converge to $(D_s - \Delta)u$ in $L^p(\mathbb{R}^{1+N})$. Hence, the set of all functions in $\mathscr{D}_p$ having compact support is dense in $\mathscr{D}_p$. On the other hand, if $u \in \mathscr{D}_p$ has compact support, a standard convolution argument shows the existence of a sequence of smooth functions with compact support converging to u in $\mathscr{D}_p$, since W is bounded in each neighborhood of the support of u. □

The next result is again proved for all $p \in [1, +\infty]$. It will allow us to control the drift term by the heat operator and the potential.

Proposition 3.4. *Let W be a function satisfying* (A2). *Then, there exists a constant $\alpha > 0$ (depending only on N, β, γ, K_β, K'_γ, c_0) such that*

$$\|W^{\frac{1}{2}} \nabla_x u\|_p \le \varepsilon\, \|(\Delta - D_s)u\|_p + \frac{\alpha}{\varepsilon}\, \|Wu\|_p, \tag{3.2}$$

for all $\varepsilon \in (0,1]$, $1 \le p \le +\infty$, and $u \in \mathscr{D}_p$.

Proof. It suffices to show the proposition for test functions u. Lemma 3.3 then allows us to extend the result to all $u \in \mathscr{D}_p$ by approximation. We further can replace W by $W + \lambda$ for some $\lambda \ge 0$ such that (A2) holds for $W + \lambda$ with $K_\beta = K'_\gamma = 0$. Since $W \ge c_0 > 0$, the estimate (3.2) for $W + \lambda$ implies (3.2) for W (with a different α). So, we may and will assume that $K_\beta = K'_\gamma = 0$ in (A2). Hence,

$$|D_s W^{-1}| \le \beta \quad \text{and} \quad |\nabla_x W^{-\frac{1}{2}}| \le \frac{\gamma}{2} \qquad \text{in } \mathbb{R}^{1+N}. \tag{3.3}$$

In what follows we write ∇ instead of ∇_x. Our arguments rely on a localization procedure in space and time. We set $\tau := \tau(s_0, x_0) = (4\beta\ell_1 W(s_0, x_0))^{-1}$ for every given $(s_0, x_0) \in \mathbb{R}^{1+N}$ and a number $\ell_1 \ge 1$ to be fixed later. Since $\tau \le (4\beta W(s_0, x_0))^{-1}$, from (3.3) it follows that

$$\frac{4}{5}\, W(s, x_0)^{-1} \le W(s_0, x_0)^{-1} \le \frac{4}{3}\, W(s, x_0)^{-1},$$

$$\frac{\sqrt{3}}{2}\,W(s,x_0)^{\frac12} \le W(s_0,x_0)^{\frac12} \le \frac{\sqrt{5}}{2}\,W(s,x_0)^{\frac12},$$

for all $t\in\mathbb{R}$ with $|t-s_0|\le\tau$. We next set $r:=r(s_0,x_0)=\sqrt{3}(2\ell_2\gamma W(s_0,x_0)^{\frac12})^{-1}$ for a number $\ell_2\ge 1$ to be chosen later. Note that $r\le(\ell_2\gamma)^{-1}W(s,x_0)^{-\frac12}$ for every $t\in(s_0-\tau,s_0+\tau)$. Estimate (3.3) then implies

$$\frac{2\ell_2-1}{2\ell_2}\,W(s,x)^{\frac12} \le W(s,x_0)^{\frac12} \le \frac{2\ell_2+1}{2\ell_2}\,W(s,x)^{\frac12},$$

for all $x\in B(x_0,r)$ and $s\in(s_0-\tau,s_0+\tau)$. We thus obtain

$$\frac{(2\ell_2-1)\sqrt{3}}{4\ell_2}\,W(s,x)^{\frac12} \le W(s_0,x_0)^{\frac12} \le \frac{(2\ell_2+1)\sqrt{5}}{4\ell_2}\,W(s,x)^{\frac12}, \tag{3.4}$$

for all (s,x) in the parabolic cylinder $Q=Q(s_0,x_0):=(s_0-\tau,s_0+\tau)\times B(x_0,r)$. We now choose functions $\eta\in C_c^\infty(\mathbb{R}^N)$ and $\zeta\in C_c^\infty(\mathbb{R})$ such that $\mathbb{1}_{B(x_0,r/2)}\le\eta\le\mathbb{1}_{B(x_0,r)}$, $\mathbb{1}_{(s_0-\tau/2,s_0+\tau/2)}\le\zeta\le\mathbb{1}_{(s_0-\tau,s_0+\tau)}$, $|\nabla\eta|\le c/r$, $|D^2\eta|\le c/r^2$ and $|D_s\zeta|\le c/\tau$ for a constant c independent of s_0, x_0, τ and r. We set $Q'=Q'(s_0,x_0):=(s_0-\tau/2,s_0+\tau/2)\times B(x_0,r/2)$ and denote the p-norms on Q' and Q by the additional indexes Q' and Q, respectively, for $1\le p\le+\infty$. Using (3.4), Lemma 3.2, the definitions of r,τ and Young's inequality, we compute

$$\begin{aligned}
&\|W^{\frac12}\nabla u\|_{p,Q'} \le c\,W(s_0,x_0)^{\frac12}\,\|\nabla u\|_{p,Q'} \le c\,W(s_0,x_0)^{\frac12}\,\|\nabla(\zeta\eta u)\|_p\\
&\quad\le c\,\|(\Delta-D_s)(\zeta\eta u)\|_p^{\frac12}\,\|W(s_0,x_0)\zeta\eta u\|_p^{\frac12}\\
&\quad\le c\,\|Wu\|_{p,Q}^{\frac12}\Big(\|(\Delta-D_s)u\|_{p,Q}+\frac1r\,\|\nabla u\|_{p,Q}+\Big(\frac1\tau+\frac1{r^2}\Big)\|u\|_{p,Q}\Big)^{\frac12}\\
&\quad\le c\,\|Wu\|_{p,Q}^{\frac12}\,\|(\Delta-D_s)u\|_{p,Q}^{\frac12}+c\gamma\,\|Wu\|_{p,Q}^{\frac12}\,\|W^{\frac12}\nabla u\|_{p,Q}^{\frac12}+c(\beta+\gamma^2)\,\|Wu\|_{p,Q}\\
&\quad\le \delta\,\|W^{\frac12}\nabla u\|_{p,Q}+\varepsilon\,\|(\Delta-D_s)u\|_{p,Q}+\frac{c(\delta)}{\varepsilon}\,\|Wu\|_{p,Q},
\end{aligned}\tag{3.5}$$

for each $\delta,\varepsilon\in(0,1]$, where the constants c only depend on N, b, ℓ_1, ℓ_2, and the last one also on δ, where b is any number such that $0<\beta,\gamma\le b$.

In the case $p=+\infty$, we fix $\ell_1=\ell_2=1$ and note that inequality (3.5) trivially yields

$$W^{\frac12}(s_0,x_0)\,|\nabla u(s_0,x_0)| \le \delta\,\|W^{\frac12}\nabla u\|_\infty+\varepsilon\,\|(\Delta-D_s)u\|_\infty+\frac{c(\delta)}{\varepsilon}\,\|Wu\|_\infty.$$

We now fix $\delta=1/2$ and take the supremum over $(s_0,x_0)\in\mathbb{R}^{1+N}$ of the left hand side. The assertion then follows.

For $p\in[1,+\infty)$, we take advantage of Proposition A.1. For this purpose, we fix the parameters ℓ_1 and ℓ_2 in the following way:

$$\ell_1=\frac{2\ell_2^2\gamma^2}{3\beta},\qquad \ell_2=\max\left\{\sqrt{3}\left(\frac12+\frac{\sqrt{\beta}}{\gamma}\right),1\right\}.$$

Clearly, $\ell_1,\ell_2\ge 1$. Moreover, since $\sqrt{\frac{\tau(s_0,x_0)}{2}}=\frac{r(s_0,x_0)}{2}$ for any $(s_0,x_0)\in\mathbb{R}^{1+N}$, the cylinder $Q'(s_0,x_0)$ coincides with the ball $B_d((s_0,x_0),\varrho(s_0,x_0))$ centered at (s_0,x_0) and with radius $\varrho(s_0,x_0):=\sqrt{3}(4\ell_2\gamma W(s_0,x_0)^{\frac12})^{-1}=\frac12 r(s_0,x_0)$, in the metric d (see (A.1) and (A.2)), whereas $Q(s_0,x_0)$ is properly contained in $B_d((s_0,x_0),2\varrho(s_0,x_0))$. Further, using (3.3) we can easily estimate

$$\begin{aligned}
&|W(s,x)^{-\frac12}-W(s_0,x_0)^{-\frac12}|\\
&\qquad\le |W(s,x)^{-\frac12}-W(s,x_0)^{-\frac12}|+|W(s,x_0)^{-\frac12}-W(s_0,x_0)^{-\frac12}|\\
&\qquad\le |W(s,x)^{-\frac12}-W(s,x_0)^{-\frac12}|+|W(s,x_0)^{-1}-W(s_0,x_0)^{-1}|^{\frac12}
\end{aligned}$$

$$\leq \frac{\gamma}{2}|x-x_0| + \sqrt{\beta}|s-s_0|^{1/2} \leq \Big(\frac{\gamma}{2}+\sqrt{\beta}\Big)\, d((s,x),(s_0,x_0)),$$

for any $(s,x),(s_0,x_0) \in \mathbb{R}^{1+N}$. The choice of ℓ_2 implies that the function ϱ is Lipschitz continuous in $\mathbb{R}^{1+N}$ with Lipschitz constant not greater than $1/4$. Hence, Proposition A.1 guarantees the existence of a countable covering $Q_k' = Q'(s_k,x_k)$ of $\mathbb{R}^{1+N}$ such each $(s,x) \in \mathbb{R}^{1+N}$ is contained in at most $K(N)$ of the cylinders $Q_k = Q(s_k,x_k)$, for some integer $K(N)$. Inequality (3.5) now implies that

$$\begin{aligned}
\|W^{\frac12}\nabla u\|_p^p &\leq \sum_{k=1}^{+\infty} \|W^{\frac12}\nabla u\|_{p,Q_k'}^p \\
&\leq 3^{p-1}\sum_{k=1}^{+\infty}\Big(\delta^p\,\|W^{\frac12}\nabla u\|_{p,Q_k}^p + \varepsilon^p\,\|(\Delta-D_s)u\|_{p,Q_k}^p + \frac{c(\delta)^p}{\varepsilon^p}\,\|Wu\|_{p,Q_k}^p\Big) \\
&\leq 3^{p-1}K(N)\Big(\delta^p\,\|W^{\frac12}\nabla u\|_p^p + \varepsilon^p\,\|(\Delta-D_s)u\|_p^p + \frac{c(\delta)^p}{\varepsilon^p}\,\|Wu\|_p^p\Big).
\end{aligned}$$

Fixing $\delta = (3K(N))^{-1}$, we get the assertion also for $p \in [1,+\infty)$. ☐

Remark 3.5. The above proof shows the following fact (cf. the remarks after (3.5)). Assume that (A2) holds for some $\beta,\gamma \in (0,b]$ with $K_\beta = K_\gamma' = 0$. Let $p = +\infty$. Then the constant α in Proposition 3.4 only depends on N and b.

Assume that (A1) holds and fix $p \in (1,+\infty)$. It is known that the realization in $L^p(\mathbb{R}^{1+N})$ of the operator $\mathrm{div}_x(a\nabla_x) - D_s$ with domain $W_p^{1,2}(\mathbb{R}^{1+N})$ has a nonempty resolvent set, cf. [20, Corollary 2.6]. This fact easily implies that there exists a constant $C_p^0 > 0$ with

$$\begin{aligned}
\frac{1}{C_p^0}\,(\|(\Delta-D_s)u\|_p + \|u\|_p) &\leq \|(\mathrm{div}_x(a\nabla_x)-D_s)u\|_p + \|u\|_p \\
&\leq C_p^0\,(\|(\Delta-D_s)u\|_p + \|u\|_p)
\end{aligned} \tag{3.6}$$

for all $u \in W_p^{1,2}(\mathbb{R}^{1+N})$.

Corollary 3.6. *Let* $1 < p < +\infty$ *and assume that* (A1)–(A4) *hold. We then have*

$$\|W^{\frac12}\nabla_x u\|_p \leq \varepsilon\,\|\mathscr{L}u\|_p + \frac{c}{\varepsilon}\,\|Wu\|_p,$$

for every $\varepsilon \in (0,\varepsilon_0]$, $u \in \mathscr{D}_p$ *and some constants* $c,\varepsilon_0 > 0$ *only depending on* C_p^0 *(see* (3.6)*) and the constants in* (A1)–(A4).

Proof. For all $u \in \mathscr{D}_p$ and $\varepsilon \in (0,1]$, Proposition 3.4 and (3.6) imply that

$$\begin{aligned}
\|W^{\frac12}\nabla_x u\|_p &\leq \varepsilon\,\|(\Delta-D_s)u\|_p + \frac{\alpha}{\varepsilon}\,\|Wu\|_p \\
&\leq c\varepsilon\,\|\mathscr{L}u - F\cdot\nabla_x u + Vu\|_p + \frac{c}{\varepsilon}\,\|Wu\|_p \\
&\leq c\varepsilon\,\|\mathscr{L}u\|_p + c\varepsilon\,\|W^{\frac12}\nabla_x u\|_p + \frac{c}{\varepsilon}\,\|Wu\|_p,
\end{aligned}$$

where the constants c only depend on the constants in (3.6) and in (A1)–(A4). The assertion follows if we take a sufficiently small $\varepsilon > 0$. ☐

We now come to the crucial *a priori* estimate.

Proposition 3.7. *Let* $p \in (1,+\infty)$. *Assume that the assumptions* (A1)–(A5) *and the inequality*

$$\frac{\theta}{p} + (p-1)\Big(\frac{\beta+\gamma\kappa}{p} + \frac{\gamma^2M^2}{4}\Big) < 1 \tag{3.7}$$

hold, where $M = \sup\{\|a(s,x)^{\frac{1}{2}}\| : (s,x) \in \mathbb{R}^{1+N}\}$*. Then, there exists a constant* $C_p > 0$ *(only depending on* C_p^0*,* M *and the constants in* (A1)–(A5)*) such that*

$$C_p^{-1}\,\|u\|_{\mathscr{D}_p} \le \|\mathscr{L}u\|_p + \|u\|_p \le C_p\,\|u\|_{\mathscr{D}_p}, \qquad u \in \mathscr{D}_p. \tag{3.8}$$

Proof. We observe that the second estimate in (3.8) follows from Proposition 3.4. Concerning the first estimate, we can restrict ourselves to the case where $K_\beta = K'_\gamma = 0$ in (A2). Indeed, in the general case it suffices to fix a large $\lambda > 0$ such that $W + \lambda$ satisfies (A2) with $K_\beta = K'_\gamma = 0$. The established estimate for the operator $\mathscr{L} - \lambda$ with the potential $V + \lambda$ then yields

$$\|u\|_{\mathscr{D}_p} \le c\,(\|\mathscr{L}u - \lambda u\|_p + \|u\|_p) \le c\,(\|\mathscr{L}u\| + \|u\|_p),$$

for some constants only depending on C_p^0, M and the constants in (A1)–(A5). Moreover, in view of Lemma 3.3, it is enough to prove the first inequality in (3.8) for test functions u .

So, let us fix $u \in C_c^\infty(\mathbb{R}^{1+N})$. At first we take $p \in [2,+\infty)$. We set $f := -\mathscr{L}u$, multiply this equality by the function $W^{p-1}|u|^{p-2}u$ and integrate by parts over $\mathbb{R}^{1+N}$. We then obtain (writing div and ∇ for, respectively, div_x and ∇_x)

$$\begin{aligned}
&\int_{\mathbb{R}^{1+N}} fu|u|^{p-2}W^{p-1}\,ds\,dx \qquad (3.9)\\
&= \frac{1}{p}\int_{\mathbb{R}^{1+N}} (D_s|u|^p)\,W^{p-1}\,ds\,dx + \int_{\mathbb{R}^{1+N}} \langle a\nabla u, \nabla(u|u|^{p-2}W^{p-1})\rangle\,ds\,dx\\
&\quad - \frac{1}{p}\int_{\mathbb{R}^{1+N}} F\cdot(\nabla|u|^p)\,W^{p-1}\,ds\,dx + \int_{\mathbb{R}^{1+N}} VW^{p-1}|u|^p\,ds\,dx\\
&= \frac{1-p}{p}\int_{\mathbb{R}^{1+N}} |u|^pW^{p-2}D_sW\,ds\,dx + (p-1)\int_{\mathbb{R}^{1+N}} \langle a\nabla u, \nabla u\rangle\,|u|^{p-2}W^{p-1}\,ds\,dx\\
&\quad + (p-1)\int_{\mathbb{R}^{1+N}} \langle a\nabla u, \nabla W\rangle\,u|u|^{p-2}W^{p-2}\,ds\,dx\\
&\quad + \frac{p-1}{p}\int_{\mathbb{R}^{1+N}} (F\cdot\nabla W)\,|u|^pW^{p-2}\,ds\,dx + \int_{\mathbb{R}^{1+N}} \Big(V + \frac{1}{p}\,\mathrm{div}\,F\Big)W^{p-1}|u|^p\,ds\,dx.
\end{aligned}$$

These equations are also valid if $p \in (1,2)$, but then the integration by parts needs some justification given by [36]. From now on we thus take $p \in (1,+\infty)$. Assumptions (A3) and (A5) further yield

$$V + \frac{1}{p}\,\mathrm{div}\;F \ge \Big(1 - \frac{\theta}{p}\Big)W. \tag{3.10}$$

Formulas (3.9) and (3.10), Hölder's inequality, and conditions (A2) and (A4) imply

$$\begin{aligned}
\Big(1-\frac{\theta}{p}\Big)\|Wu\|_p^p &+ (p-1)\int_{\mathbb{R}^{1+N}} |a^{\frac{1}{2}}\nabla u|^2|u|^{p-2}W^{p-1}\,ds\,dx\\
&\le \|f\|_p\,\|Wu\|_p^{p-1} + \frac{\beta(p-1)}{p}\,\|Wu\|_p^p + \frac{\gamma\kappa(p-1)}{p}\,\|Wu\|_p^p\\
&\quad + (p-1)\int_{\mathbb{R}^{1+N}} |a^{\frac{1}{2}}\nabla u|\,|a^{\frac{1}{2}}\nabla W|\,|u|^{p-1}W^{p-2}\,ds\,dx.
\end{aligned}$$

Using again (A2) and Hölder's inequality, the last summand can be estimated by

$$\begin{aligned}
&(p-1)\gamma M\int_{\mathbb{R}^{1+N}} |a^{\frac{1}{2}}\nabla u|\,W^{p-\frac{1}{2}}|u|^{p-1}\,ds\,dx\\
&\le (p-1)\gamma M\Big(\int_{\mathbb{R}^{1+N}} W^p|u|^p\,ds\,dx\Big)^{\frac{1}{2}}\Big(\int_{\mathbb{R}^{1+N}} |a^{\frac{1}{2}}\nabla u|^2|u|^{p-2}W^{p-1}\,ds\,dx\Big)^{\frac{1}{2}}\\
&=: (p-1)\gamma M\,AB.
\end{aligned}$$

By means of Young's inequality, we then deduce

$$\Big[1-\frac{\theta}{p}-\frac{p-1}{p}(\beta+\gamma\kappa)-\varepsilon\Big]A^2+(p-1)B^2-(p-1)\gamma MAB\le c(\varepsilon)\|f\|_p^p,\quad (3.11)$$

for each $\varepsilon>0$ and some $c(\varepsilon)>0$. Due to assumption (3.7), we can fix $\varepsilon>0$ such that the left hand side of (3.11) is larger than $\eta(A^2+B^2)$ for some $\eta>0$. So, we have shown that

$$\|Wu\|_p^p+\int_{\mathbb{R}^{1+N}}|a^{\frac12}\nabla u|^2|u|^{p-2}W^{p-1}\,ds\,dx\le c\,\|f\|_p^p=c\,\|\mathscr{L}u\|_p^p.\qquad (3.12)$$

Here and below the constants c only depend on M, C_p^0 (see (3.6)) and the constants in (A1)–(A5). Assumption (A4), Corollary 3.6 and the estimate (3.12) further yield

$$\|F\cdot\nabla u\|_p\le\kappa\,\|W^{\frac12}\nabla u\|_p\le c\,(\|\mathscr{L}u\|_p+\|Wu\|_p)\le c\,\|\mathscr{L}u\|_p\,.$$

Using (3.6), the last inequality, (3.12) and recalling that $V\le c_1W$, we get

$$\|u\|_{\mathscr{D}_p}\le c\,(\|\mathscr{L}u-F\cdot\nabla u+Vu\|_p+\|Wu\|_p)\le c\,\|\mathscr{L}u\|_p\,,$$

which is the remaining part of (3.8). □

We now want to treat the inhomogeneous parabolic equation

$$D_su(s)=\mathrm{div}_x(a(s)\nabla_xu(s))+F(s)\cdot\nabla_xu(s)-V(s)u(s)+f(s),\quad s\in\mathbb{R},\quad (3.13)$$

on $\mathbb{R}^N$. For this purpose, we define the operator $L_pu=\mathscr{L}u$ with $D(L_p)=\mathscr{D}_p$ in $L^p(\mathbb{R}^{1+N})$, where $1<p<+\infty$. In the next theorem we identify $L^p(\mathbb{R}^{1+N})$ with $L^p(\mathbb{R},L^p(\mathbb{R}^N))$ and we use the following concepts. An *evolution family* $G(s,r)$, $s\ge r$, is a family of bounded operators on a Banach space X such that

$$G(t,s)G(s,r)=G(t,r),\quad G(s,s)=I,\quad (s,r)\mapsto G(s,r)\text{ is strongly continuous},$$

for $r,s,t\in\mathbb{R}$ with $t\ge s\ge r$. The corresponding *evolution semigroup* on $L^p(\mathbb{R},X)$ is given by

$$(S(t)f)(s)=G(s,s-t)f(s-t),$$

for $f\in L^p(\mathbb{R}^{1+N})$, $s\in\mathbb{R}$ and $t\ge0$. (See e.g. [13] or [40].)

Theorem 3.8. *Let $p\in(1,+\infty)$ and assume that conditions* (A1)–(A5) *and* (3.7) *are satisfied. Then, the following assertions hold.*

(a) The operator L_p generates a positive and contractive evolution semigroup $S_p(\cdot)$ on $L^p(\mathbb{R}^{1+N})$ induced by an evolution family $G_p(s,r)$, $s\ge r$, of positive contractions on $L^p(\mathbb{R}^N)$.

(b) We set $u:=G_p(\cdot,r)\varphi$ for every $\varphi\in L^p(\mathbb{R}^N)$ and $r\in\mathbb{R}$. Then, $u\in W_p^{1,2}((a,b)\times\mathbb{R}^N)$, $Vu\in L^p((a,b)\times\mathbb{R}^N)$ and $D_su(s)=\mathscr{A}(s)u(s)$ for $s\in(a,b)$ and each interval $[a,b]\subset(r,+\infty)$. Moreover, for each $f\in L^p(\mathbb{R}^{1+N})$ there exists a unique $u\in\mathscr{D}_p$ satisfying (3.13)*, namely*

$$u(s)=-L_p^{-1}f(s)=\int_{-\infty}^sG_p(s,r)f(r)\,dr,\qquad s\in\mathbb{R}.$$

(c) Let conditions (A5) *and* (3.7) *also hold for some $q\in(1,+\infty)$. Then, $S_p(\cdot)$ and $S_q(\cdot)$ (resp. $G_p(\cdot,\cdot)$ and $G_q(\cdot,\cdot)$) coincide in $L^p(\mathbb{R}^{1+N})\cap L^q(\mathbb{R}^{1+N})$ (resp. in $L^p(\mathbb{R}^N)\cap L^q(\mathbb{R}^N)$).*

Proof. Being rather long, we split the proof in three steps.

Step 1. Due to Proposition 3.7, the operator L_p is closed on $\mathscr{D}_p$. Hence, Lemma 3.3 implies that the space $C_c^\infty(\mathbb{R}^{1+N})$ is a core for L_p. The dissipativity of L_p now follows from Lemma 3.1. As a result, L_p generates a contraction semigroup on $L^p(\mathbb{R}^{1+N})$ if $I-L_p$ is invertible, thanks to the Lumer–Phillips theorem (see e.g., [22, Theorem II.3.15]). We employ the operators $\mathscr{L}_\tau=\mathrm{div}_x(a\nabla_x)+\tau F\cdot\nabla_x-V-D_s$

for $\tau \in [0,1]$. Since these operators satisfy (A1)–(A5) and (3.7) with the same constants, Proposition 3.7 combined with the dissipativity of $\mathscr{L}_\tau$ yield that

$$\|u\|_{\mathscr{D}_p} \le c\,(\|\mathscr{L}_\tau u\|_p + \|u\|_p) \le c\,(\|\mathscr{L}_\tau u - u\|_p + \|u\|_p) \le c\,\|u - \mathscr{L}_\tau u\|_p,$$

for every $u \in \mathscr{D}_p$, with constants independent of $\tau \in [0,1]$. Hence, $I - L_p$ is invertible if $I - \mathscr{L}_0 : \mathscr{D}_p \to L^p(\mathbb{R}^{1+N})$ is invertible, see e.g., [27, Theorem 5.2]. Observe that $\mathscr{L}_0$ has no drift term. We use the Yosida approximations $V_\varepsilon = V(1+\varepsilon V)^{-1}$ and $W_\varepsilon = W(1+\varepsilon W)^{-1}$ of V and W, respectively, where $\varepsilon \in (0,1]$. It is easy to check that the potential W_ε and the coefficients of $\mathscr{L}_{0,\varepsilon} = \operatorname{div}_x(a\nabla_x) + V_\varepsilon - D_s$ also satisfy (A1)–(A5) and (3.7) with the same constants (except that one has to replace c_0 by $c_0(1+c_0)^{-1}$). Moreover, $\mathscr{L}_{0,\varepsilon}$ with domain $W^{1,2}_p(\mathbb{R}^{1+N})$ generates a contraction semigroup on $L^p(\mathbb{R}^{1+N})$. For every $f \in L^p(\mathbb{R}^{1+N})$ and $\varepsilon \in (0,1]$, we can thus define $u_\varepsilon = (I - \mathscr{L}_{0,\varepsilon})^{-1} f \in W^{1,2}_p(\mathbb{R}^{1+N})$, i.e., $u_\varepsilon - \mathscr{L}_{0,\varepsilon} u_\varepsilon = f$. From the dissipativity of $\mathscr{L}_{0,\varepsilon}$ and Proposition 3.7 we deduce that $\|u_\varepsilon\|_p \le \|f\|_p$ and

$$\|W_\varepsilon u_\varepsilon\|_p + \|u_\varepsilon\|_{W^{1,2}_p(\mathbb{R}^{1+N})} \le c\,(\|\mathscr{L}_{0,\varepsilon} u_\varepsilon\|_p + \|u_\varepsilon\|_p) \le c\,\|f\|_p\,,$$

where the constants c do not depend on $\varepsilon \in (0,1]$. So, we find a sequence (u_{ε_n}) converging weakly to a function $u \in W^{1,2}_p(\mathbb{R}^{1+N})$. A subsequence converges in $L^p_{\mathrm{loc}}(\mathbb{R}^{1+N})$, so that we may assume that $u_{\varepsilon_n} \to u$ a.e. in $\mathbb{R}^{1+N}$. This fact implies that $\|Wu\|_p \le c\,\|f\|_p$, and hence $u \in \mathscr{D}_p$. Finally, we can pass to the limit (in the sense of distributions) in the equation $u_\varepsilon - \mathscr{L}_{0,\varepsilon} u_\varepsilon = f$, obtaining $u - \mathscr{L}_0 u = f$. Consequently, $I - \mathscr{L}_0$ with domain $\mathscr{D}_p$ is invertible so that L_p generates a contraction semigroup $S(\cdot)$ on $L^p(\mathbb{R}^{1+N})$.

Let us now check that $T_p(\cdot)$ is an evolution semigroup and that the associated evolution operator $G_p(\cdot,\cdot)$ is contractive. For this purpose. we begin by noting that $\mathscr{D}_p$ is a dense subset of $C_0(\mathbb{R}, L^p(\mathbb{R}^N))$ and $(I - L_p)^{-1}$ is continuous from $L^p(\mathbb{R}, L^p(\mathbb{R}^N))$ into $C_0(\mathbb{R}, L^p(\mathbb{R}^N))$. Moreover,

$$L_p(\varphi f) = \varphi L_p f - \varphi' f,$$

for all $f \in \mathscr{D}_p$ and $\varphi \in C^1_c(\mathbb{R})$. Theorem 3.4 of [38] now yields the existence of an evolution family $G_p(s,r)$, $s \ge r$, such that $(S_p(t)f)(s) = G_p(s, s-t) f(s-t)$ for $f \in L^p(\mathbb{R}^{1+N})$, $s \in \mathbb{R}$, and $t \ge 0$ (see also [40, Theorem 4.2] and the references therein). By [38, Formula (3.3)], for all $s > r$ it holds that $\|G_p(s,r)\|_{L(L^p(\mathbb{R}^N))} \le \|T_p(s-r) S_0(r-s)\|_{L(L^p(\mathbb{R}^{1+N}))}$, where $S_0(\cdot)$ is the semigroup of left translations (i.e., $S_0(t)f = f(\cdot - t)$ for $t \in \mathbb{R}$ and $f \in L^p(\mathbb{R}^{1+N})$). Since both $T_p(\cdot)$ and $S_0(\cdot)$ are contractive semigroups, the contractivity of the operator $G_p(s,r)$ on $L^p(\mathbb{R}^N)$ for all $s \ge r$ follows at once.

Step 2. By Step 1, the operator $\delta I - L_p$ is invertible for all $\delta > 0$. On the other hand, for sufficiently small $\delta \in (0, c_0)$ also the operator $\mathscr{L} + \delta I$ satisfies assumptions (A1)–(A5) and (3.7) for the potentials $V - \delta$ and $W - \delta$, different constants $c_0, c_1, K_\beta, K'_\gamma$ and slightly increased α, β, θ and κ. As a consequence, also the operator L_p is invertible, whence the second part of assertion (b) follows. (Use [13, p. 68] for the formula for L_p^{-1}.)

Let now fix $\varphi \in L^p(\mathbb{R}^N)$, $r \in \mathbb{R}$, and $[a,b] \subset (r, +\infty)$. Take a function $\phi \in C^\infty_c(\mathbb{R})$ with $\phi \equiv 1$ on $[a,b]$ and support contained in $(r, +\infty)$. Define the function $u \in L^p(\mathbb{R}, L^p(\mathbb{R}^N))$ by $u(s) = \phi(s) G(s,r)\varphi$ for $s \ge r$ and $u(s) = 0$ for $s < r$. As in [13, p. 64] one sees that $u \in D(L_p) = \mathscr{D}_p$ and $L_p u(s) = -\phi'(s) G(s,r)\varphi$ for $s \ge r$. So, we have established assertion (b).

Step 3. It remains to show part (c) and the asserted positivity in (a). Theorem 3.4 of [35] states that the operator $A_p(s) = (\mathscr{A}(s), W^2_p(\mathbb{R}^N) \cap D(W(s)))$ generates a contraction semigroup $(e^{tA_p(s)})_{t \ge 0}$ on $L^p(\mathbb{R}^N)$ for each $s \in \mathbb{R}$. Moreover,

$A_p(s)$ admits $C_c^\infty(\mathbb{R}^N)$ as a core. This semigroup is positive because of Theorems 3.3 and 4.1 of [3], see also [35, Proposition 6.1]. As in [22, Paragraph III.4.13], one verifies that the multiplication operator $A_p(\cdot)$ with maximal domain

$$D(A_p(\cdot)) = \{u \in L^p(\mathbb{R}^{1+N}) : u(s) \in D(A_p(s)) \text{ for a.e. } s \in \mathbb{R},\ A_p(\cdot)u \in L^p(\mathbb{R}^{1+N})\}$$

generates the semigroup $M(\cdot)$ on $L^p(\mathbb{R}^{1+N})$ given by $M(t)f = e^{tA_p(\cdot)}f(\cdot)$, which is positive and contractive. Moreover, the first derivative $-D_s$ with domain $W_p^1(\mathbb{R}, L^p(\mathbb{R}^N))$ generates the positive contraction semigroup $S_0(\cdot)$ on $L^p(\mathbb{R}^{1+N})$. Observe that $D(A_p(\cdot)) \cap D(-D_s) = \mathscr{D}_p$ and $L_p = A_p(\cdot) - D_s$. Therefore, the Lie-Trotter product formula (see [22, Corollary III.5.8]) implies the positivity of $S(t)$, and thus of $G(s,r)$, for all $t \geq 0$ and $s \geq r$. The semigroups $M(\cdot)$ and $S_0(\cdot)$ on $L^p(\mathbb{R}^{1+N})$ for different values of p coincide on the intersections of the L^p spaces (see [3, Theorem 3.3] or [35, Lemma 4.3]). Hence, the Lie–Trotter product formula further shows that the respective evolution semigroups, and thus the evolution families, coincide. □

In the following remark we indicate that Theorem 3.8 cannot be deduced from known results in the autonomous case.

Remark 3.9. Under the assumptions of Theorem 3.8 we define the operator $A(s)$ in $L^p(\mathbb{R}^N)$ by setting $A(s)\varphi = \mathscr{A}(s)\varphi$ for $\varphi \in D(A(s)) := \{v \in W_p^2(\mathbb{R}^N) : W(s)v \in L^p(\mathbb{R}^N)\}$, $s \in \mathbb{R}$ and $p \in (1, +\infty)$. Theorem 3.4 and Proposition 6.5 of [35] then state that the operators $A(s)$ are sectorial and have maximal L^p-regularity (with uniform constants). We refer the reader to [28] for the concept of maximal L^p-regularity. In addition, assume for a moment that the operators $A(s)$ satisfy the Acquistapace–Terreni condition; i.e., that there are constants $L \geq 0$ and $\mu, \nu \in (0,1]$ such that $\mu + \nu > 1$ and

$$\|\lambda^\nu A(t)(\lambda - A(t))^{-1}(A(t)^{-1} - A(s)^{-1})\| \leq L\,|t-s|^\mu \tag{3.14}$$

holds for all $\lambda > 0$ and $t, s \in \mathbb{R}$, see [1, 2]. Corollary 2.6 in [20] then implies that for some $\omega \geq 0$ the operator $A(\cdot) - D_s - \omega I$ with domain $\mathscr{D}_p$ is invertible in $L^p(\mathbb{R}^{1+N})$. We point out that this fact is the crucial point of the proof of Theorem 3.8. However, the Acquistapace–Terreni condition does not follow from the assumptions of Theorem 3.8, as we now show by a simple example.

Let $a = I$, $F = 0$, $N = 1$, $p = 2$, and set $W(s,x) = V(s,x) = \exp(\exp(s+x))$ for $(s,x) \in \mathbb{R}^2$. It is easy to check that the assumptions (A1)–(A5) and (3.7) hold in this case. On the other hand, if (3.14) were true, then $D(A(0)) = W_2^2(\mathbb{R}^2) \cap D(V(0))$ would be contained in the real interpolation space $(X, D(A(s)))_{\nu,\infty}$ which is embedded into $D(V(s)^\alpha)$ for all $s \in \mathbb{R}$ and $\alpha \in (0,\nu)$. (See e.g. [31] for basic facts on interpolation theory.). Given such an $\alpha \in (0,\nu)$ take $s > 0$ such that $\alpha e^s = 2$. Choose a function $\chi \in C^2(\mathbb{R})$ which vanishes on $\mathbb{R}_-$ and is equal to 1 on $[1, +\infty)$. Set $v(x) = \chi(x)\exp(-\frac{3}{2}e^x)$ for $x \in \mathbb{R}$. It is straightforward to verify that $v \in D(A(0))$ but $v \notin D(V(s)^\alpha)$, so that (3.14) has to be violated in this example.

In order to apply Theorem 3.8 to the parabolic Ornstein–Uhlenbeck operator $\mathscr{G}_O$, we have to study the mapping properties of the isomorphism $M_p : L^p(\mathbb{R}^{1+N}) \to L^p(\mathbb{R}^{1+N}, \nu)$, see (2.9), on the space

$$\mathscr{D}_{p,O} = \{u \in W_p^{1,2}(\mathbb{R}^{1+N}) : |x|^2 u \in L^p(\mathbb{R}^{1+N})\},$$

endowed with the norm $\|u\|_{\mathscr{D}_{p,O}} = \|u\|_{W_p^{1,2}(\mathbb{R}^{1+N})} + \|\,|x|^2 u\|_p$, i.e., the space $\mathscr{D}_p$ for the potential $W_O(x) = c_0 + k_1\,|x|^2$ from (2.16).

Lemma 3.10. *Assume that Hypothesis 2.1 holds and let $p \in (1, +\infty)$. Then, the map M_p defined in* (2.9) *induces an isomorphism from $\mathscr{D}_{p,O}$ onto $W_p^{1,2}(\mathbb{R}^{1+N}, \nu)$.*

Proof. We have to prove that the restrictions $M_p : \mathscr{D}_{p,O} \to W^{1,2}_p(\mathbb{R}^{1+N},\nu)$ and $M_p^{-1} : W^{1,2}_p(\mathbb{R}^{1+N},\nu) \to \mathscr{D}_{p,O}$ are well-defined and continuous. Concerning M_p, it suffices to show

$$\|M_p u\|_{W^{1,2}_p(\mathbb{R}^{1+N},\nu)} \le c\,(\|u\|_{W^{1,2}_p(\mathbb{R}^{1+N})} + \|\,|x|^2\,u\|_p), \tag{3.15}$$

for a constant c and all $u \in C_c^\infty(\mathbb{R}^{1+N})$, because of Lemma 3.3. We further recall that the norm of the functions $|x|\,|\nabla_x u|$ in $L^p(\mathbb{R}^{1+N})$ can be controlled by the norm of u in $\mathscr{D}_{p,O}$, due to Corollary 3.6. The formulas (2.11), (2.12) and (2.14) combined with Lemma 2.2 now easily imply (3.15).

To establish the continuity of the operator $M_p^{-1} : W^{1,2}_p(\mathbb{R}^{1+N},\nu) \to \mathscr{D}_{p,O}$ we first note that the space $C_c^\infty(\mathbb{R}^{1+N})$ is dense in $W^{1,2}_p(\mathbb{R}^{1+N},\nu)$. This fact can be shown as in Lemma 3.3. It remains to prove that

$$\|M_p^{-1}v\|_{W^{1,2}_p(\mathbb{R}^{1+N})} + \|\,|x|^2\,M_p^{-1}v\|_p \le c\,\|v\|_{W^{1,2}_p(\mathbb{R}^{1+N},\nu)}, \tag{3.16}$$

for a constant c and all $v \in C_c^\infty(\mathbb{R}^{1+N})$. For the derivatives of $u := M_p^{-1}v$ one can obtain expressions similar to those in (2.11). Hence, we have to dominate the norms in $L^p(\mathbb{R}^{1+N})$ of the functions $|x|u$, $|x|^2u$ and $|x|\,|M_p^{-1}(D_iv)|$ by $\|v\|_{W^{1,2}_p(\mathbb{R}^{1+N},\nu)}$. We prove below that there exists $c>0$ such that

$$\begin{aligned}\int_{\mathbb{R}^{1+N}} |x|^p|M_p^{-1}v|^p\,ds\,dx &\le c\int_{\mathbb{R}^{1+N}} |x|^p|v|^p\,d\nu\\ &\le c\sum_{j=1}^N\int_{\mathbb{R}^{1+N}}|D_jv|^p\,d\nu + c\int_{\mathbb{R}^{1+N}}|v|^p\,d\nu.\end{aligned} \tag{3.17}$$

After (3.17) has been shown, we can apply this inequality also to the functions D_iv and x_iv, where $i=1,\dots,N$. In this way we derive (3.16).

To show (3.17), let v be a test function. At first, Lemma 2.2 yields

$$\begin{aligned}\int_{\mathbb{R}^{1+N}} |x|^p|M_p^{-1}v|^p\,ds\,dx &= (2\pi)^{\frac{N}{2}}\int_{\mathbb{R}^{1+N}}|x|^p|v|^p(\det Q_s)^{\frac12}\,d\nu\\ &\le (2\pi C_2)^{\frac{N}{2}}\int_{\mathbb{R}^{1+N}}|x|^p|v|^p\,d\nu.\end{aligned}$$

To check the second part of (3.17), we first deduce from Lemma 2.2 the estimates

$$\begin{aligned}\int_{\mathbb{R}^{1+N}}|x|^p|v|^p\,d\nu &\le (2\pi C_1)^{-\frac{N}{2}}\int_{\mathbb{R}^{1+N}}|x|^p|v|^p\,e^{-\frac12\langle Q_s^{-1}x,x\rangle}\,ds\,dx\\ &\le \frac{C_2^p}{(2\pi C_1)^{\frac{N}{2}}}\int_{\mathbb{R}^{1+N}}|v|^p|Q_s^{-1}x|^p\,e^{-\frac12\langle Q_s^{-1}x,x\rangle}\,ds\,dx.\end{aligned}$$

On the other hand, [35, Lemma 7.1] implies that

$$\int_{\mathbb{R}^N}|v(s,x)|^p|Q_s^{-1}x|^pe^{-\frac12\langle Q_s^{-1}x,x\rangle}dx \le c\int_{\mathbb{R}^N}(|v(s,x)|^p+|\nabla_xv(s,x)|^p)e^{-\frac12\langle Q_s^{-1}x,x\rangle}dx,$$

for a constant $c>0$ and every $s\in\mathbb{R}$. We integrate this inequality with respect to $s\in\mathbb{R}$ and use once more Lemma 2.2. As a result, (3.17) holds. □

We come now to our second main result which describes the domain of the parabolic Ornstein-Uhlenbeck operator $\mathscr{G} = \mathscr{A}_O(\cdot) - D_s$ in the Lebesgue space with the family of invariant measures. This fact has immediate consequences on the regularity properties of the equation (1.6).

Theorem 3.11. *Let $p\in(1,+\infty)$ and assume that Hypothesis 2.1 holds. Then, the operator $G_p = (\mathscr{G}, W^{1,2}_p(\mathbb{R}^{1+N},\nu))$ generates a positive contraction semigroup $T(\cdot)$ on $L^p(\mathbb{R}^{1+N})$. This semigroup is given by $(T(t)f)(s) = G_O(s,s-t)f(s-t)$ for $f\in L^p(\mathbb{R}^{1+N},\nu)$, $s\in\mathbb{R}$, $t\ge0$, and the positive and contractive evolution family*

$G_O(s,r)$, $s \geq r$, solving (1.1)*. Further, $u := G_O(\cdot,r)\varphi \in W_p^{1,2}((a,b)\times\mathbb{R}^N,\nu)$ and $D_t u(s) = \mathscr{A}_O(s)u(s)$ for every $\varphi \in L^p(\mathbb{R}^N)$, $r \in \mathbb{R}$, $[a,b] \subset (r,+\infty)$ and $s \in (a,b)$. Finally, for each $f \in L^p(\mathbb{R}^{1+N},\nu)$ there exists a unique $u \in D(G_p)$ satisfying* (1.6)*, namely $u := (I - G_p)^{-1} f$.*

Proof. We can apply Theorem 3.8 to the operator $\mathscr{L}_O - \lambda I$, see (2.10) and (2.16). Theorem 3.8 and Lemma 3.10 thus imply that the operator $G_p - \lambda I$ with domain $D(G_p) = M_p(\mathscr{D}_p) = W_p^{1,2}(\mathbb{R}^{1+N},\nu)$ generates a positive semigroup on $L^p(\mathbb{R}^{1+N},\nu)$. Moreover, G_p extends the operator $\mathscr{G}$ defined on test functions. As mentioned in the proof of Lemma 3.10, test functions are dense in $W_p^{1,2}(\mathbb{R}^{1+N},\nu)$ and thus they are a core for G_p. In view of [30], G_p then generates the evolution semigroup $T(\cdot)$ corresponding to G_O, as described in the introduction. (Note that Hypothesis 1.1(iv) in [30] is needed only to guarantee the continuity of the function $G(s,r)f$ with respect to r, when $f \in C_b(\mathbb{R}^N)$ and $G(s,r)$ is the evolution operator associated with the class of nonautonomous Kolmogorov operators therein considered; in our situation that assumption is not needed since the continuity of the function $G_O(s,r)f$ with respect to r is clear since we have an explicit formula for this function, see [15].) This semigroup is contractive. The remaining assertions can be shown as in Theorem 3.8. □

4. Operators with dominating potential for $p = 1, +\infty$.

In this section we extend Theorem 3.8 to the borderline cases $p = 1, +\infty$. We set $L_1 = (\mathscr{L}, \mathscr{D}_1)$ on $L^1(\mathbb{R}^{1+N}) = L^1(\mathbb{R}, L^1(\mathbb{R}^N))$ and $L_\infty = (\mathscr{L},\mathscr{D}_\infty)$ on $C_0(\mathbb{R}^{1+N}) = C_0(\mathbb{R}, C_0(\mathbb{R}^N))$. Note that in these cases we cannot expect to replace $\mathscr{D}_1$ and $\mathscr{D}_\infty$ with the intersection $W_p^{1,2}(\mathbb{R}^{1+N}) \cap D(W)$ and $W_\infty^{1,2}(\mathbb{R}^{1+N}) \cap D(W)$, respectively. To avoid some technical problems, we restrict ourselves to the case of the Laplacian, where $a(s) = I$ for all $s \in \mathbb{R}$.

We need in the next proof some properties of the operator $\Delta - D_s$ on $L^1(\mathbb{R}, L^1(\mathbb{R}^N))$. Consider the semigroup $G(\cdot)$ generated by the Laplacian on $L^1(\mathbb{R}^N)$ and let $(V(t)f)(s) = G(t)f(s-t)$ on $L^1(\mathbb{R}, L^1(\mathbb{R}^N))$ be the induced evolution semigroup, which is positive and contractive. The generator H of the semigroup $V(\cdot)$ is the closure of $\Delta - D_s$ defined on the intersection of the domains of Δ and of D_s in $L^1(\mathbb{R}, L^1(\mathbb{R}^N))$, see e.g., [13, Remark 2.35]. The semigroup $V(\cdot)$ leaves invariant the Schwartz space of rapidly decreasing functions $f : \mathbb{R}^{1+N} \to \mathbb{R}$ which, thus, is a core of H. In view of Lemma 3.3, it follows that

$$D(H) = \{u \in L^1(\mathbb{R}, L^1(\mathbb{R}^N)) : (\Delta - D_s)u \in L^1(\mathbb{R}, L^1(\mathbb{R}^N))\} =: D(\Delta - D_s).$$

We further have

$$(I - H)^{-1} f(t) = \int_{-\infty}^{t} e^{s-t} G(t-s) f(s)\, ds,$$

for all $t \in \mathbb{R}$ and $f \in L^1(\mathbb{R}, L^1(\mathbb{R}^N))$, and hence

$$D(H) \hookrightarrow W_1^{\rho-\sigma}(\mathbb{R}, W_1^{2\sigma}(\mathbb{R}^N)),$$

for $\frac{1}{2} < \sigma < \rho < 1$ and the usual Slobodeckiĭ spaces. This fact follows for bounded time intervals from [18, Theorem 19] and [19, Theorem 4]. The extension to the time interval $\mathbb{R}$ can be done as in [31, Chapter 4]. Let $a < b$ and $R > 0$. Due to Sobolev's embedding theorem the space $W_1^{2\sigma}(B(0,R))$ is embedded into $W_p^1(B(0,R))$ for some $p > 1$. Hence, $W_1^{2\sigma}(B(0,R))$ is compactly embedded into $L^p(B(0,R)) \hookrightarrow L^1(B(0,R))$. Corollary 2 in [41] now implies that $\overset{\circ}{W}{}_1^{\rho-\sigma}((a,b), W_1^{2\sigma}(B(0,R)))$ is compactly embedded into $L^1((a,b)\times B(0,R))$.

Theorem 4.1. *Assume that $a = I$ and that conditions* (A2)–(A5) *are satisfied for some $\beta,\gamma > 0$, $\theta < 1$ and $p = 1$. Then, the following assertions hold.*

(a) The operator L_1 generates a positive and contractive evolution semigroup $S(\cdot)$ on $L^1(\mathbb{R}^{1+N})$ induced by an evolution family $G(s,r)$, $s \geq r$, of positive contractions on $L^1(\mathbb{R}^N)$.

(b) We set $u := G(\cdot, r)\varphi$ for every $\varphi \in L^1(\mathbb{R}^N)$ and $r \in \mathbb{R}$. Then, the functions $(\Delta - D_s)u$ and Vu belong to $L^1((a,b) \times \mathbb{R}^N)$ and $D_s u(s) = \mathscr{A}(s)u(s)$ for all $s \in (a,b)$ and each interval $[a,b] \subset (r, +\infty)$. Moreover, for each $f \in L^1(\mathbb{R}^{1+N})$ there exists a unique function $u \in \mathscr{D}_1$ satisfying (3.13)*, namely*

$$u(s) = -L_p^{-1} f(s) = \int_{-\infty}^{s} G(s,r) f(r)\, dr, \qquad s \in \mathbb{R}.$$

(c) In addition, assume that condition (3.7) *holds for some $p \in (1, +\infty)$. Then, the evolution semigroups and evolution operators obtained in the present theorem and in Theorem 3.8 coincide on the intersection of the L^1- and L^p-spaces.*

Proof. Take $u \in C_c^\infty(\mathbb{R}^{1+N})$ and set $f = -\mathscr{L}u$. We multiply this equation by $\operatorname{sign} u$. Integrating by parts and using the dissipativity of $\Delta - D_s$ on $L^1(\mathbb{R}^{1+N})$, we then obtain

$$\begin{aligned}
&\int_{\mathbb{R}^{1+N}} (V + \operatorname{div}_x F)|u|\, ds\, dx \\
&\qquad \leq \int_{\mathbb{R}^{1+N}} (D_s - \Delta)u \operatorname{sign} u\, ds\, dx + \int_{\mathbb{R}^{1+N}} (Vu - F \cdot \nabla_x u) \operatorname{sign} u\, ds\, dx \\
&\qquad = \int_{\mathbb{R}^{1+N}} f \operatorname{sign} u\, ds\, dx \leq \|f\|_1 .
\end{aligned}$$

Assumptions (A3) and (A5) thus imply

$$(1-\theta)\, \|Wu\|_1 \leq \|\mathscr{L}u\|_1. \tag{4.1}$$

Taking into account Proposition 3.4 and proceeding as in the proof of Proposition 3.7 after estimate (3.12), we obtain the inequalities (3.8) also for $p = 1$ with constants only depending on the constants in (A2)–(A5). Hence, L_1 is closed. Moreover, the dissipativity of L_1 follows from Lemmas 3.1 and 3.3.

We want to show the invertibility of $I - L_1$. Here, we may assume that $F \equiv 0$ since the general case is then deduced by means of the continuity method as in the proof of Theorem 3.8. We use the notation introduced in that proof. Observe that the operator $\mathscr{L}_{0,\varepsilon} = \Delta - V_\varepsilon - D_s$ with domain $D(\Delta - D_s)$ generates a contraction semigroup on $L^1(\mathbb{R}^{1+N})$ for each $\varepsilon \in (0,1]$, thanks to the bounded perturbation theorem applied to the generator $\Delta - D_s$. As a consequence, for each $f \in L^1(\mathbb{R}^{1+N})$ there exists a function $u_\varepsilon \in L^1(\mathbb{R}^{1+N})$ such that $u_\varepsilon - \mathscr{L}_{0,\varepsilon} u_\varepsilon = f$. The dissipativity and (4.1) now imply

$$\|u_\varepsilon\|_1 \leq \|f\|_1, \qquad \|W_\varepsilon u_\varepsilon\|_1 \leq c\, \|f\|_1, \tag{4.2}$$

with a constant c independent of ε since V_ε and W_ε satisfy the assumptions (A1)-(A5) with uniform constants. It follows that

$$\|(\Delta - D_s)u_\varepsilon\|_1 = \|\mathscr{L}_{0,\varepsilon} u_\varepsilon + V_\varepsilon u_\varepsilon\|_1 \leq (2+c)\|f\|_1.$$

By the observations made above the statement of the theorem, there exists a null sequence (ε_n) such that $u_n := u_{\varepsilon_n}$ converges to a function u in $L^1_{\mathrm{loc}}(\mathbb{R}^{1+N})$. We infer from (4.2) that $\|u\|_1 \leq \|f\|_1$ and $\|Wu\|_1 \leq c\, \|f\|_1$. Moreover, $(\Delta - D_s)u_n \to (\Delta - D_s)u$ in $L^1_{\mathrm{loc}}(\mathbb{R}^{1+N})$ and, therefore, $\mathscr{L}u = f$ and $u \in \mathscr{D}_1$. Thus, L_1 generates a contraction semigroup. The other assertions can now be shown as in Theorem 3.8. $\square$

We now come to the space C_0. In the proof of the next result we have to estimate the oscillation of V itself, and thus we cannot work with the auxiliary potential W.

Proposition 4.2. *Assume that $a = I$ and that conditions* (A2) *and* (A4) *hold for every $\beta, \gamma > 0$ and with $W = V$. Then, there exists a constant $C_\infty > 0$ (only depending on the constants in* (A2) *and* (A4)*) such that*

$$C_\infty^{-1}\,\|u\|_{\mathscr{D}_\infty} \le \|\mathscr{L}u\|_\infty + \|u\|_\infty \le C_\infty\,\|u\|_{\mathscr{D}_\infty},$$

for any $u \in \mathscr{D}_\infty$.

Proof. The second estimate in the assertion is a consequence of Proposition 3.4. Lemma 3.3 then shows that it is enough to prove the other inequality for all test functions u. At first we assume that (A2) holds with $K_\beta = K'_\gamma = 0$ for some $\beta, \gamma \in (0, 1]$ to be fixed below.

Let $u \in C_c^\infty(\mathbb{R}^{1+N})$. Set $f = \mathscr{L}u$. Again we write ∇ instead of ∇_x. Fix $(s_0, x_0) \in \mathbb{R}^{1+N}$. As in the proof of Proposition 3.4 (with $\ell_1 = 1$ and $\ell_2 = \ell$), we define $Q = Q(s_0, x_0) = (s_0 - \tau, s_0 + \tau) \times B(x_0, r)$ and $Q' = Q'(s_0, x_0) = (s_0 - \frac{\tau}{2}, s_0 + \frac{\tau}{2}) \times B(x_0, \frac{r}{2})$ with $\tau = (4\beta V(s_0, x_0))^{-1}$ and $r = \sqrt{3}(2\ell\gamma V(s_0, x_0)^{\frac{1}{2}})^{-1}$. Here, we fix $\ell \ge 1$ such that

$$\frac{3(2\ell - 1)^2}{4(2\ell)^2} \ge \frac{2}{3} \qquad \text{and} \qquad \frac{5(2\ell + 1)^2}{4(2\ell)^2} \le \frac{3}{2}\,.$$

The inequalities (3.4) (with $W = V$) now imply that

$$\frac{2}{3}\,V(s, x) \le V(s_0, x_0) \le \frac{3}{2}\,V(s, x) \quad \text{and} \quad |V(s, x) - V(s_0, x_0)| \le \frac{1}{2}\,V(s, x), \tag{4.3}$$

for all $(s, x) \in Q$. We choose functions $\eta \in C_c^\infty(\mathbb{R}^N)$ and $\zeta \in C_c^\infty(\mathbb{R})$ such that $\mathbb{1}_{B(x_0, r/2)} \le \eta \le \mathbb{1}_{B(x_0, r)}$ and $\mathbb{1}_{(s_0 - \tau/2, s_0 + \tau/2)} \le \zeta \le \mathbb{1}_{(s_0 - \tau, s_0 + \tau)}$, $|\nabla\eta| \le c/r$, and $|D^2\eta| \le c/r^2$ and $|D_s\zeta| \le c/\tau$. Here and below the constants $c = c(\eta, \zeta)$ do not depend on s_0, x_0, τ and r. We have

$$\begin{aligned}
&\Delta(\zeta\eta u) + F \cdot \nabla(\zeta\eta u) - D_s(\zeta\eta u) - V(s_0, x_0)\zeta\eta u \\
&\quad = \zeta\eta f + u(\Delta - D_s)(\zeta\eta) + 2\zeta\nabla u \cdot \nabla\eta + \zeta u F \cdot \nabla\eta + (V - V(s_0, x_0))\zeta\eta u =: w.
\end{aligned}$$

Since $V(s_0, x_0) > 0$, the dissipativity of $\Delta + F \cdot \nabla - D_s$ on $C_c^\infty(\mathbb{R}^{1+N})$ (see Lemma 3.1) yields

$$\begin{aligned}
\|V(s_0, x_0)\zeta\eta u\|_{\infty,Q} \le \|w\|_\infty \le &\|f\|_{\infty,Q} + \Big(\frac{c}{r^2} + \frac{c}{\tau}\Big)\|u\|_{\infty,Q} + \frac{c}{r}\|\nabla u\|_{\infty,Q} \\
&+ \frac{c\kappa}{r}\|V^{\frac{1}{2}}u\|_{\infty,Q} + \|(V - V(s_0, x_0))u\|_{\infty,Q}\,,
\end{aligned}$$

where we have also used (A4) and have denoted the sup norm on Q by $\|\cdot\|_{\infty,Q}$. From (4.3) and the definition of τ and r, we then deduce

$$\frac{2}{3}\,\|Vu\|_{\infty,Q'} \le \|f\|_\infty + c_1(\gamma^2 + \kappa\gamma + \beta)\,\|Vu\|_{\infty,Q} + \gamma c_1\,\|V^{\frac{1}{2}}\nabla u\|_{\infty,Q} + \frac{1}{2}\,\|Vu\|_{\infty,Q},$$

where c_1 only depends on η and ζ. Letting (s_0, x_0) vary in $\mathbb{R}^{1+N}$, we obtain

$$\frac{2}{3}\,\|Vu\|_\infty \le \|f\|_\infty + c_1(\gamma^2 + \kappa\gamma + \beta)\,\|Vu\|_\infty + \gamma c_1\,\|V^{\frac{1}{2}}\nabla u\|_\infty + \frac{1}{2}\,\|Vu\|_\infty\,.$$

We fix $\beta = \min\{1, (18c_1)^{-1}\}$ and take $\gamma \le \gamma_0$ where $\gamma_0 \in (0, 1]$ satisfies $c_1(\gamma_0^2 + \kappa\gamma_0) \le 18^{-1}$. It then follows that

$$\|Vu\|_\infty \le 18\,\|f\|_\infty + 18\gamma_0 c_1\|V^{\frac{1}{2}}\nabla u\|_\infty. \tag{4.4}$$

Now (A4) and the equation $\mathscr{L}u = f$ imply

$$\|(\Delta - D_s)u\|_\infty \le c_2\,(\|f\|_\infty + \|V^{\frac{1}{2}}\nabla u\|_\infty), \tag{4.5}$$

for $c_2 := \max\{19, \kappa + 18\gamma_0 c_1\}$. Combining Proposition 3.4 with (4.4) and (4.5), we arrive at

$$\|V^{\frac{1}{2}}\nabla u\|_\infty \le \varepsilon\|(\Delta - D_s)u\|_\infty + \frac{\alpha}{\varepsilon}\|Vu\|_\infty$$

$$\leq c(\varepsilon)\|f\|_\infty + \Big(\varepsilon c_2 + \frac{18\alpha\gamma c_1}{\varepsilon}\Big)\|V^{\frac{1}{2}}\nabla u\|_\infty\,,$$

for all $\varepsilon \in (0,1]$. Because of Remark 3.5, the constant α is independent of γ varying in bounded sets. Finally, we set $\varepsilon = \gamma^{\frac{1}{2}}$ and $\gamma = \min\{\gamma_0, (2(c_2+18\alpha c_1))^{-2}\}$. This leads to the estimate $\|V^{\frac{1}{2}}\nabla u\|_\infty \leq c\,\|f\|_\infty$ for a constant only depending on N and κ. Inequalities (4.4) and (4.5) now yield $\|u\|_{\mathscr{D}_\infty} \leq c\,\|\mathscr{L}u\|_\infty$.

It remains to remove the restriction that $K_\beta = K'_\gamma = 0$. Above we have fixed $\beta, \gamma > 0$ depending only on N and κ. There exists a number $\lambda = \lambda(\beta,\gamma) \geq 0$ such that $V + \lambda$ satisfies (A2) for the fixed value of β and γ with $K_\beta = K'_\gamma = 0$. Hence, the first estimate in the assertion holds for $V+\lambda$ and all test functions u. It then holds for V itself with a possibly larger constant C_∞. □

As before Theorem 4.1, one can verify that $(V(t)f)(s) = G(t)f(s-t)$ defines a positive contraction semigroup on $C_0(\mathbb{R}, C_0(\mathbb{R}^N))$ whose generator is given by $\Delta - D_s$ on its maximal domain.

Theorem 4.3. *Assume that $a = I$, that $V \in C(\mathbb{R}^{1+N})$ and that* (A2)–(A4) *are satisfied for all $\beta, \gamma > 0$. Then, the following assertions hold.*

(a) The operator $L_\infty = (\mathscr{L}, \mathscr{D}_\infty)$ generates a positive and contractive evolution semigroup $S(\cdot)$ on $C_0(\mathbb{R}^{1+N})$ induced by an evolution family $G(s,r)$, $s \geq r$, of positive contractions on $C_0(\mathbb{R}^N)$.

(b) We set $u := G(\cdot, r)\varphi$ for every $\varphi \in C_0(\mathbb{R}^N)$ and $r \in \mathbb{R}$. Then, the functions $(\Delta - D_s)u$ and Vu belong to $C([a,b], C_0(\mathbb{R}^N))$ and $D_s u(s) = \mathscr{A}(s)u(s)$ for all $s \in (a,b)$ and each interval $[a,b] \subset (r,+\infty)$. Moreover, for each $f \in C_0(\mathbb{R}^{1+N})$ there exists a unique function $u \in \mathscr{D}_\infty$ satisfying (3.13)*, namely*

$$u(s) = -L_p^{-1}f(s) = \int_{-\infty}^{s} G(s,r)f(r)\,dr, \qquad s \in \mathbb{R}.$$

(c) If also the assumptions of Theorems 3.8 or 4.1 hold for some $p \in [1,+\infty)$, then the evolution semigroup and the evolution family obtained in the present theorem and in Theorems 3.8 or 4.1, respectively, coincide on the intersection of the C_0- and L^p-spaces.

Proof. We first show that L_∞ generates a contraction semigroup on $C_0(\mathbb{R}^{1+N})$ in the case when $V = W$. Lemmas 3.1 and 3.3 and Proposition 4.2 show that L_∞ is closed, densely defined and dissipative. Moreover, as in the proof of Theorem 3.8, we can restrict ourselves to the case $F \equiv 0$ since Proposition 4.2 gives a suitable a priori estimate. We use the notation introduced in that proof. Replacing V by $V+\lambda$ we can suppose that $K_\beta = K'_\gamma = 0$. We fix $p > N+2$ and sufficiently small $\beta, \gamma > 0$ such that (3.7) hold for this p, $M = 1$ and $\theta = \kappa = 0$. Observe that $W^{1,2}_p(\mathbb{R}^{1+N}) \hookrightarrow C_0(\mathbb{R}, C^1_0(\mathbb{R}^N))$. For each $f \in C_c(\mathbb{R}^{1+N})$ and each $\varepsilon \in (0,1]$, there exists a function $u_\varepsilon \in W^{1,2}_p(\mathbb{R}^{1+N})$ such that

$$u_\varepsilon - (\Delta - V_\varepsilon - D_s)u_\varepsilon = f, \tag{4.6}$$

since V_ε is a bounded perturbation of the generator $\Delta - D_s$. By dissipativity, we have $\|u_\varepsilon\|_r \leq \|f\|_r$ for $r = p, +\infty$. Propositions 3.7 and 4.2 also yield

$$\|(\Delta - D_s)u_\varepsilon\|_r + \|V_\varepsilon u_\varepsilon\|_r \leq c\,\|f\|_r,$$

for $r = p, +\infty$. Here and below the constants c do not depend on ε. Since the sequence (u_ε) is bounded in $W^{1,2}_p(\mathbb{R}^{1+N})$, which continuously embeds in $C^\alpha(\mathbb{R}^{1+N})$ for a suitable $\alpha > 0$, the Arzelà-Ascoli theorem implies that u_{ε_n} converges locally uniformly in $\mathbb{R}^{1+N}$ to a function u, for a suitable null sequence (ε_n). Due to (4.6), also $(\Delta - D_s)u_{\varepsilon_n}$ converges uniformly on compact sets so that

$$u - (\Delta - D_s)u + Vu = f \quad \text{ and } \quad \|Vu\|_r + \|(\Delta - D_s)u\|_r \leq c\,\|f\|_r,$$

for $r = p, +\infty$. Therefore, $u \in W_p^{1,2}(\mathbb{R}^{1+N}) \hookrightarrow C_0(\mathbb{R}, C_0^1(\mathbb{R}^N))$. We next show that Vu belongs to $C_0(\mathbb{R}^{1+N})$. Take $(s_0, x_0) \in \mathbb{R}^{1+N}$ and $\eta \in C_c^\infty(\mathbb{R}^{1+N})$ such that $\mathbb{1}_{B((s_0,x_0),R)} \le \eta \le \mathbb{1}_{B((s_0,x_0),2R)}$, $\|\nabla\eta\|_\infty \le c/R$ and $\|D^2\eta\|_\infty \le c/R^2$ for all $R \ge 1$. Then

$$\eta u - (\Delta - D_s)(\eta u) + V\eta u = \eta f - 2\nabla_x\eta \cdot \nabla_x u - u\Delta\eta + uD_s\eta$$

and Proposition 4.2 (applied to $\eta u \in \mathscr{D}_\infty$) shows that

$$\begin{aligned}|V(s_0,x_0)u(s_0,x_0)| &\le \|V\eta u\|_\infty \\ &\le c\Big[\|\eta f\|_\infty + \|\eta u\|_\infty + \frac{1}{R}\|\nabla_x u\|_\infty + \frac{1}{R}\|u\|_\infty\Big] \\ &\le c\Big[\|f\|_{L^\infty(B((s_0,x_0),R))} + \|u\|_{L^\infty(B((s_0,x_0),R))} + \frac{1}{R}\|\nabla_x u\|_\infty + \frac{1}{R}\|u\|_\infty\Big].\end{aligned}$$

Fix $\varepsilon > 0$ and let R be sufficiently large such that $R^{-1}(\|u\|_\infty + \|\nabla_x u\|_\infty) \le \varepsilon$. Further, let $M > R$ be so large such that $|f(s,x)| + |u(s,x)| \le \varepsilon$ for any $|(s,x)| \ge M$ (this is possible since $u, f \in C_0(\mathbb{R}^{1+N})$). The above inequality implies that $|Vu(s_0,x_0)u(s_0,x_0)| \le 2c\varepsilon$ if $|(s_0,x_0)| \ge M + R$, so that $Vu \in C_0(\mathbb{R}^{1+N})$. We conclude that $(\Delta - D_s)u = u + Vu - f \in C_0(\mathbb{R}^{1+N})$ and $u \in \mathscr{D}_\infty$. As a consequence, $I - L_\infty$ has dense range and thus L_∞ (also with $F \ne 0$) generates a contraction semigroup, provided that $V = W$. Given $0 \le f \in C_0(\mathbb{R}^{1+N})$ and $\lambda > 0$, there is a function $u \in \mathscr{D}_\infty$ with $u - L_\infty u = f$. If u were not non-negative, it would have a strictly negative minimum. This fact would easily lead to a contradiction. Hence, L_∞ has a positive resolvent and generates a positive semigroup.

Now, let V be as in the statement. For $0 \le \tau \le 1$, we introduce the potential $V_\tau = W + \tau(V - W)$ and the operator $\mathscr{L}_\tau = \Delta + F\cdot\nabla - V_\tau - D_s$ with $D(\mathscr{L}_\tau) = \mathscr{D}_\infty$. Observe that $V_0 = W \le V_\tau \le V_1 = V$. We know that $\mathscr{L}_0$ generates a positive contraction semigroup on $C_0(\mathbb{R}^{1+N})$. Whenever also $\mathscr{L}_\tau$ generates a contraction semigroup $(e^{t\mathscr{L}_\tau})_{t\ge0}$, we can apply the Lie-Trotter formula to the sum $\mathscr{L}_\tau = \mathscr{L}_0 + W - V_\tau$ to derive that $0 \le e^{t\mathscr{L}_\tau} \le e^{t\mathscr{L}_0}$ for all $t \ge 0$, see [22, Corollary III.5.8]. Since $(I - \mathscr{L}_\tau)^{-1}$ and $(I - \mathscr{L}_0)^{-1}$ are, respectively, the Laplace transform of $e^{t\mathscr{L}_\tau}$ and $e^{t\mathscr{L}_0}$ at $\lambda = 1$, we obtain $0 \le (I - \mathscr{L}_\tau)^{-1} \le (I - \mathscr{L}_0)^{-1}$ and, using also (A3),

$$0 \le (V_\sigma - V_\tau)(I - \mathscr{L}_\tau)^{-1} \le (c_1 - 1)(\sigma - \tau)W(I - \mathscr{L}_0)^{-1}$$

for all $0 \le \tau \le 1$ such that $\mathscr{L}_\tau$ generates a positive contraction semigroup and for all $\sigma \in [\tau, 1]$. On the other hand, Proposition 4.2 implies that $\|W(I - \mathscr{L}_0)^{-1}\| \le 3C_\infty$. By finitely many perturbation steps of the form $\mathscr{L}_\sigma = \mathscr{L}_\tau + V_\tau - V_\sigma$, we can then conclude that $\mathscr{L}_1 = L_\infty$ generates a positive contraction semigroup on $C_0(\mathbb{R}^{1+N})$. The remaining assertions can be shown as in Theorem 3.8. ☐

Appendix A. A variant of the Besicovitch covering theorem

In this appendix, we prove a variant of the classical Besicovitch covering theorems, in which balls are replaced by cylinders. This proposition plays a crucial role in the proof of Proposition 3.4.

Let us introduce the distance d on $\mathbb{R}^{1+N}$ defined by

$$d((t,x),(s,y)) = \max\{|t-s|^{1/2}, |x-y|\}, \qquad (t,x),\ (s,y) \in \mathbb{R}^{1+N}. \tag{A.1}$$

A straightforward computation shows that d is in fact a metric which defines the same topology in $\mathbb{R}^{1+N}$ as the Euclidean norm $|\cdot|$. Moreover, $(\mathbb{R}^{1+N}, d)$ and $(\mathbb{R}^{1+N}, |\cdot|)$ have the same bounded sets. For all $(s_0,x_0) \in \mathbb{R}^{1+N}$ and $r > 0$, we denote by $B_d((s_0,x_0),r)$ the ball with center at (s_0,x_0) and radius r in the metric d. Note that

$$B_d((s_0,x_0),r) = (s_0 - r^2, s_0 + r^2) \times B(x_0, r). \tag{A.2}$$

We can now state and prove the following proposition.

Proposition A.1. *Let $\varrho : \mathbb{R}^{1+N} \to (0,+\infty)$ be a bounded Lipschitz continuous function (with respect to the distance d) with Lipschitz constant $\kappa < 1$, i.e.,*

$$|\varrho(s,x) - \varrho(r,y)| \le \kappa d((s,x),(r,y)), \qquad (s,x),\ (r,y) \in \mathbb{R}^{1+N}.$$

Then, there exists a sequence $((s_n,x_n)) \subset \mathbb{R}^{1+N}$ such that the family $\mathscr{F} = \{B_d((s_n,x_n);\varrho(s_n,x_n)) : n \in \mathbb{N}\}$ is a covering of $\mathbb{R}^{1+N}$. Moreover, for each $\lambda \in [1,\kappa^{-1})$ there exists a number $\zeta(\kappa,\lambda,N)$ such that every subset $I \subset \mathbb{N}$ with $\bigcap_{n\in I} B_d((s_n,x_n),\lambda\varrho(s_n,x_n)) \neq \varnothing$ contains at most $\zeta(\kappa,\lambda,N)$ elements.

Proof. We adapt partly the proof of the classical Besicovitch covering theorem given in [21, Section 1.5.2, Theorem 2] to our situation. Being rather long, we split the proof into several steps.

Step 1. Let us set

$$\delta = \sup\{\varrho(s,x) :\ (s,x) \in \mathbb{R}^{1+N}\},$$

and define the sets

$$\begin{aligned} A^{(l)} &= \{(s,x) \in \mathbb{R}^{1+N} :\ \omega(l-1) \le d((s,x),(0,0)) \le \omega l\} \\ \delta_1^{(l)} &= \max\{\varrho(s,x) : (s,x) \in A^{(l)}\}, \qquad l \in \mathbb{N}, \end{aligned}$$

where ω is a positive constant greater than $2\kappa^{-1}\delta$. For each $l \in \mathbb{N}$, we are going to construct a countable family of balls $\mathcal{F}^{(l)} = \{B_d((s_n^{(l)},x_n^{(l)}),\varrho(s_n^{(l)},x_n^{(l)})) : n \in \mathbb{N}\}$, which, as we will show in the forthcoming steps, will represent a countable covering of the set $A^{(l)}$. The family $\mathscr{F}$ we are looking for will be then defined as the union of all the balls from the families $\mathscr{F}^{(l)}$ $(l \in \mathbb{N})$.

We set $A_1^{(l)} := A^{(l)}$. Let us fix $l \in \mathbb{N}$ and an arbitrary point $(s_1^{(l)},x_1^{(l)}) \in A_1^{(l)}$ such that $\varrho(s_1^{(l)},x_1^{(l)}) \ge \frac{3}{4}\delta_1^{(l)}$. Next, we consider the set $A_2^{(l)} := A_1^{(l)} \setminus B_d((s_1^{(l)},x_1^{(l)}),\varrho(s_1^{(l)},x_1^{(l)}))$, set $\delta_2^{(l)} := \max\{\varrho(s,x) : (s,x) \in A_2^{(l)}\}$, and we pick up an arbitrary point $(s_2^{(l)},x_2^{(l)}) \in A_2^{(l)}$ such that $\varrho(s_2^{(l)},x_2^{(l)}) \ge \frac{3}{4}\delta_2^{(l)}$. We then inductively define the sequence $(s_n^{(l)},x_n^{(l)})$ in this way: $(s_m^{(l)},x_m^{(l)})$ is any arbitrary fixed point in $A_m^{(l)} := A_{m-1}^{(l)} \setminus B_d((s_{m-1}^{(l)},x_{m-1}^{(l)}),\varrho(s_{m-1}^{(l)},x_{m-1}^{(l)}))$ such that $\varrho(s_m^{(l)},x_m^{(l)}) \ge 3\delta_m^{(l)}/4$, where $\delta_m^{(l)} = \max\{\varrho(s,x) : (s,x) \in A_m^{(l)}\}$.

We have two possibilities: either there exists $m \in \mathbb{N}$ such that $A_{m+1}^{(l)} = \varnothing$ or $A_n \neq \varnothing$ for all $n \in \mathbb{N}$. In the first case, we set $I^{(l)} = \{1,\ldots,m_0^{(l)}\}$, where $m_0^{(l)}$ is the smallest integer such that $A_{m_0^{(l)}+1}^{(l)} = \varnothing$. In the second case, we set $I^{(l)} = \mathbb{N}$.

Let $\lambda > 0$. In the sequel, to simplify the notation, we set

$$B_{i,\lambda}^{(1)} := B_d((s_i^{(l)},x_i^{(l)}),\lambda\varrho(s_i^{(l)},x_i^{(l)})), \qquad B_i^{(l)} := B_{i,1}^{(l)}, \qquad \varrho_i^{(l)} := \varrho(s_i^{(l)},x_i^{(l)}). \quad \text{(A.3)}$$

Step 2. Here, for every $l \in \mathbb{N}$, we prove that the balls $B_{i,1/3}^{(l)}$ $(i \in I^{(l)})$ are all disjoint. For this purpose we first observe that $\varrho_i^{(l)} \ge \frac{3}{4}\varrho_j^{(l)}$ if $j > i$. Indeed,

$$\varrho_i^{(l)} \ge \frac{3}{4}\max\{\varrho(s,x) : (s,x) \in A_i^{(l)}\} \ge \frac{3}{4}\max\{\varrho(s,x) : (s,x) \in A_j^{(l)}\} \ge \frac{3}{4}\varrho_j^{(l)},$$

since $A_i \supset A_j$.

Using this inequality, we can now prove that the balls $B_{i,1/3}^{(l)}$ $(i \in I^{(l)})$ are all disjoint. Assume, by contradiction, that there exists $(s,y) \in B_{i,1/3}^{(l)} \cap B_{j,1/3}^{(l)}$ for some indexes i and j. Then, the triangle inequality yields

$$\begin{aligned} d((s_i^{(l)},x_i^{(l)}),(s_j^{(l)},x_j^{(l)})) &\le d((s_i^{(l)},x_i^{(l)}),(s,y)) + d((s,y),(s_j^{(l)},x_j^{(l)})) \\ &\le \frac{1}{3}\varrho_i^{(l)} + \frac{1}{3}\varrho_j^{(l)} \le \frac{1}{3}\varrho_i^{(l)} + \frac{4}{9}\varrho_i^{(l)} = \frac{7}{9}\varrho_i^{(l)}. \end{aligned}$$

As a result, $(s_j^{(l)}, x_j^{(l)}) \in B_i^{(l)}$. This is impossible since, by construction, the point $(s_j^{(l)}, x_j^{(l)})$ belongs to the set $A_j^{(l)}$ which is contained in the complement of $B_i^{(l)}$.

Step 3. Here, we show for the case $I^{(l)} = \mathbb{N}$ that the sequence $(\varrho_n^{(l)})$ tends to 0 as $n \to +\infty$. As we have already noticed, $(s_m^{(l)}, x_m^{(l)}) \notin B_n^{(l)}$ if $m > n$. Hence,

$$d((s_m^{(l)}, x_m^{(l)}), (s_n^{(l)}, x_n^{(l)})) \geq \varrho_n^{(l)} = \frac{1}{3}\varrho_n^{(l)} + \frac{2}{3}\varrho_n^{(l)} \geq \frac{1}{3}\varrho_n^{(l)} + \frac{1}{2}\varrho_m^{(l)} \geq \frac{1}{3}\left(\varrho_n^{(l)} + \varrho_m^{(l)}\right). \tag{A.4}$$

Since $(s_n^{(l)}, x_n^{(l)}) \in A^{(l)}$ for any $n \in \mathbb{N}$, the sequence $((s_n^{(l)}, x_n^{(l)}))$ is bounded with respect to the distance d and, by the remarks at the very beginning of the section, it is bounded with respect to the Euclidean norm as well. Thus, there exists a subsequence $(t_{n_k}^{(l)}, x_{n_k}^{(l)})$ which converges with respect to the Euclidean norm (and, hence, with respect to the distance d) to a point $(s, x) \in A_1^{(l)}$. From (A.4), it follows that the sequence $(\varrho_{n_k}^{(l)})$ tends to 0 as $k \to +\infty$. The same arguments can then be used to prove that any subsequence of $(\varrho_n^{(l)})$ has a subsequence which converges to 0. Hence, $\varrho_n^{(l)}$ tends to 0 as $n \to +\infty$, as well.

Step 4. We can now prove that, for each $l \in \mathbb{N}$, the family $\mathscr{F}^{(l)}$ is a covering of the set $A^{(l)}$. Of course, we have only to consider the case when $I^{(l)} = \mathbb{N}$. So, let us fix a point $(s^*, x^*) \in A^{(l)}$. Since, by Step 3, the sequence $(\varrho_n^{(l)})$ vanishes as $n \to +\infty$, we can fix $n_0 \geq 2$ such that $\varrho_{n_0}^{(l)} < 3\varrho(s^*, x^*)/4$. This implies that $(s^*, x^*) \in B_j^{(l)}$ for some $j \leq n_0 - 1$. Indeed, if this were not the case, then $(s^*, x^*) \in A_{n_0}^{(l)}$ and, hence,

$$\varrho_{n_0}^{(l)} \geq \frac{3}{4}\max\{\varrho(s,x) : (s,x) \in A_{n_0}\} \geq \frac{3}{4}\varrho(s^*, x^*),$$

a contradiction.

Step 5. Here, we prove that, for every $l \in \mathbb{N}$ and every $\lambda \in [1, \kappa^{-1})$, there exists $\xi(\kappa, \lambda, N)$ such that any ball of the family $\mathscr{F}_\lambda^{(l)} := \{B_{i,\lambda}^{(l)} : i \in I^{(l)}\}$ intersects at most $\xi(\kappa, \lambda, N)$ other balls of the family. Here, $B_{i,\lambda}^{(l)}$ is defined by (A.3). As a byproduct, we then deduce that, if $J \subset I^{(l)}$ is a finite set of indexes such that $\bigcap_{i \in J} B_{i,\lambda}^{(l)} \neq \varnothing$, then J contains at most $\xi(\kappa, \lambda, N)$ elements.

Let us fix a ball $B_{i_0,\lambda}^{(l)}$ and let J be a finite set of indexes such that $B_{i,\lambda}^{(l)} \cap B_{i_0,\lambda}^{(l)} \neq \varnothing$ for every $i \in J$. Clearly,

$$d((s_{i_0}^{(l)}, x_{i_0}^{(l)}), (s_i^{(l)}, x_i^{(l)})) \leq \lambda\left(\varrho_{i_0}^{(l)} + \varrho_i^{(l)}\right).$$

Since, by assumptions the function ϱ is κ-Lipschitz continuous, we have

$$|\varrho_{i_0}^{(l)} - \varrho_i^{(l)}| \leq \kappa d((s_{i_0}^{(l)}, x_{i_0}^{(l)}), (s_i^{(l)}, x_i^{(l)})).$$

Hence,

$$|\varrho_{i_0}^{(l)} - \varrho_i^{(l)}| \leq \kappa\lambda\left(\varrho_{i_0}^{(l)} + \varrho_i^{(l)}\right)$$

or, equivalently,

$$\varrho_{i_0}^{(l)} \leq \frac{\kappa\lambda + 1}{1 - \kappa\lambda}\varrho_i^{(l)}, \qquad \varrho_i^{(l)} \leq \frac{\kappa\lambda + 1}{1 - \kappa\lambda}\varrho_{i_0}^{(l)}. \tag{A.5}$$

We now observe that for all $i \in J$ and $(s, x) \in B_{i,1/3}^{(l)}$ it holds that

$$\begin{aligned} d((s,x), (s_{i_0}^{(l)}, x_{i_0}^{(l)})) &\leq d((s,x), (s_i^{(l)}, x_i^{(l)})) + d((s_i^{(l)}, x_i^{(l)}), (s_{i_0}^{(l)}, x_{i_0}^{(l)})) \\ &\leq \frac{1}{3}\varrho_i^{(l)} + \lambda\left(\varrho_{i_0}^{(l)} + \varrho_i^{(l)}\right) = \left(\frac{1}{3} + \lambda\right)\varrho_i^{(l)} + \lambda\varrho_{i_0}^{(l)} \end{aligned}$$

$$\leq \left\{ \left(\frac{1}{3} + \lambda \right) \frac{\kappa\lambda + 1}{1 - \kappa\lambda} + \lambda \right\} \varrho_{i_0}^{(l)} = \frac{\kappa\lambda + 6\lambda + 1}{3 - 3\kappa\lambda} \varrho_{i_0}^{(l)}.$$

Therefore, $B_{i,1/3}^{(l)} \subset B_{i_0,\sigma_\kappa}^{(l)}$ for every $i \in J$, where $\sigma_\kappa := (\kappa\lambda + 6\lambda + 1)/(3 - 3\kappa\lambda)$. Now, recalling that the balls $B_{i,1/3}^{(l)}$ ($i \in I^{(l)}$) are all disjoint, it follows that

$$3^{-N-2} 2\omega_N \sum_{i \in J} (\varrho_i^{(l)})^{N+2} = m\left(\bigcup_{i \in J} B_{i,1/3}^{(l)} \right)$$
$$\leq m(B_{i_0,\sigma_\kappa}^{(l)}) = 2\omega_N \left(\frac{\kappa\lambda + 6\lambda + 1}{3 - 3\kappa\lambda} \right)^{N+2} (\varrho_{i_0}^{(l)})^{N+2}, \quad \text{(A.6)}$$

where m and ω_N denote, respectively, the Lebesgue measure in $\mathbb{R}^N$ and the Lebesgue measure of the unit ball in $\mathbb{R}^N$. Using (A.5) we can estimate

$$\sum_{i \in J} (\varrho_i^{(l)})^{N+2} \geq \mathrm{card}(J) \left(\frac{1 - \kappa\lambda}{\kappa\lambda + 1} \right)^{N+2} (\varrho_{i_0}^{(l)})^{N+2}. \quad \text{(A.7)}$$

From (A.6) and (A.7) we now get

$$3^{-N-2} \mathrm{card}(J) \left(\frac{1 - \kappa\lambda}{\kappa\lambda + 1} \right)^{N+2} (\varrho_{i_0}^{(l)})^{N+2} \leq \left(\frac{\kappa\lambda + 6\lambda + 1}{3 - 3\kappa\lambda} \right)^{N+2} (\varrho_{i_0}^{(l)})^{N+2},$$

i.e.,

$$\mathrm{card}(J) \leq \xi(\kappa, \lambda, N) := \left[\left(\frac{\kappa^2\lambda^2 + 2\kappa\lambda(1 + 3\lambda) + 6\lambda + 1}{(1 - \kappa\lambda)^2} \right)^{N+2} \right],$$

where $[\,\cdot\,]$ denotes the integer part of the quantity in brackets.

Step 6. We now prove that, for every $l \in \mathbb{N}$ and every $\lambda \in [1, \kappa^{-1})$, the intersection of more than $\zeta(\kappa, \lambda, N) := 2\xi(\lambda, \kappa, N) + 2$ balls from the family $\mathscr{F}_\lambda := \{B_{i,\lambda}^{(l)} : l \in \mathbb{N},\, i \in I^{(l)}\}$ is empty. For this purpose, we reorder each family $\mathscr{F}_\lambda^{(l)} := \{B_{i,\lambda}^{(l)} : i \in I^{(l)}\}$ ($l \in \mathbb{N}$) into the union of $\xi(\kappa, \lambda, N) + 1$ subfamilies of disjoint balls. Let us fix $l \in \mathbb{N}$ and define the function $\sigma_\lambda^{(l)} : \mathbb{N} \to \{1, \ldots, \xi(\kappa, \lambda, N) + 1\}$ inductively as follows. For $j = 1, \ldots, \xi(\kappa, \lambda, N) + 1$, we set $\sigma_\lambda^{(l)}(j) = j$. Take an integer $m > \xi(\kappa, \lambda, N) + 1$. Suppose $\sigma_\lambda^{(l)}(j)$ is defined for every $j \in \{1, \ldots, m\}$. Let us define $\sigma_\lambda^{(l)}(m+1)$. For this purpose, we introduce the set $\mathcal{H}_{\lambda,m}^{(l)} = \{j = 1, \ldots, m : B_{j,\lambda}^{(l)} \cap B_{m+1,\lambda}^{(l)} \neq \varnothing\}$. By Step 5, $\mathcal{H}_{\lambda,m}^{(l)}$ has less than $\xi(\kappa, \lambda, N) + 1$ elements. Hence, there exists the minimal $h_m \in \{1, \ldots, \xi(\kappa, \lambda, N) + 1\}$ such that $h_m \notin \sigma_\lambda^{(l)}(\mathcal{H}_{\lambda,m}^{(l)})$. Then we have $B_{r,\lambda}^{(l)} \cap B_{m+1,\lambda}^{(l)} = \varnothing$ for all $r \in \{1, \ldots, m\}$ satisfying $\sigma_\lambda^{(l)}(r) = h_m$. We define $\sigma_\lambda^{(l)}(m+1) := h_m$.

Let us now set $\mathscr{G}_{h,\lambda}^{(l)} := \{B_{i,\lambda}^{(l)} : \sigma_\lambda^{(l)}(i) = h\}$ for each $h \in \{1, \ldots, \xi(\kappa, \lambda, N) + 1\}$. From the very definition of the function $\sigma_\lambda^{(l)}$, the set $\mathscr{G}_{h,\lambda}^{(l)}$ consists of disjoint balls. Clearly, each ball of the family $\mathscr{F}_\lambda^{(l)}$ belongs to $\mathscr{G}_{h,\lambda}^{(l)}$ for a (unique) $h \in \mathbb{N}$. So we have split the family $\mathscr{F}_\lambda^{(l)}$ into the union of the families $\mathscr{G}_{j,\lambda}^{(l)}$ ($j = 1, \ldots, \xi(\kappa, \lambda, N) + 1$).

We now introduce the sets $\mathscr{G}_{j,\lambda}$ ($j = 1, \ldots, \zeta(\kappa, \lambda, N)$) defined as follows:

$$\mathscr{G}_{j,\lambda} = \bigcup_{l=1}^{+\infty} \mathscr{G}_{j,\lambda}^{(2l-1)}, \qquad j = 1, \ldots, \xi(\kappa, \lambda, N) + 1,$$
$$\mathscr{G}_{j,\lambda} = \bigcup_{l=1}^{+\infty} \mathscr{G}_{j-\xi(\kappa,\lambda,N),\lambda}^{(2l)}, \qquad j = \xi(\kappa, \lambda, N) + 2, \ldots \zeta(\kappa, \lambda, N).$$

Note that every family $\mathscr{G}_{j,\lambda}$ consists of disjoint balls. Indeed, suppose that B_1 and B_2 belong to $\mathscr{G}_{j,\lambda}$ for some j and $B_1 \cap B_2 \neq \varnothing$. (We assume that $j \leq \xi(\kappa, \lambda, N) + 1$ but the same argument can be applied in the case when $j > \xi(\kappa, \lambda, N) + 1$.) Then, $B_1 \in \mathscr{G}_{j,\lambda}^{(2l_1-1)}$ and $B_2 \in \mathscr{G}_{j,\lambda}^{(2l_2-1)}$ for some $l_1, l_2 \in \mathbb{N}$. Clearly, from the above results $l_1 \neq l_2$ and, without loss of generality, we can assume that $l_1 < l_2$. Denote by (s_1, x_1) and (s_2, x_2) the centers of the balls B_1 and B_2, respectively. Since $B_1 \cap B_2 \neq \varnothing$, we have

$$d((s_1, x_1), (s_2, x_2)) \leq \lambda\,(\varrho(s_1, x_1) + \varrho(s_2, x_2)) \leq \lambda\,(\delta + \delta) = 2\lambda\delta.$$

On the other hand, $(s_1, x_1) \in A^{(2l_1-1)}$ and $(s_2, x_2) \in A^{(2l_2-1)}$. Hence,

$$\begin{aligned} d((s_1, x_1), (s_2, x_2)) &\geq d((s_2, x_2), (0,0)) - d((s_1, x_1), (0,0)) \\ &\geq \omega(2l_2 - 2) - \omega(2l_1 - 1) \\ &= \omega(2(l_2 - l_1) - 1) \geq \omega, \end{aligned}$$

which leads us to a contradiction, since $\omega > 2\kappa^{-1}\delta$. It is now clear that

$$\sum_{j=1}^{\zeta(\kappa,\lambda,N)} \sum_{B_{i,\lambda}^{(l)} \in \mathscr{G}_{j,\lambda}} \chi_{B_{i,\lambda}^{(l)}}(s, x) \leq \zeta(\kappa, \lambda, N), \qquad (s, x) \in \mathbb{R}^{1+N},$$

and this completes the proof. □

References

[1] P. Acquistapace: *Evolution operators and strong solutions of abstract linear parabolic equations*, Differential Integral Equations **1** (1988), 433–457.

[2] P. Acquistapace, B. Terreni: *A unified approach to abstract linear nonautonomous parabolic equations*, Rend. Sem. Mat. Univ. Padova **78** (1987), 47–107.

[3] W. Arendt, G. Metafune, D. Pallara: *Schrödinger operators with unbounded drift,* J. Operator Theory **55** (2006), 185–211.

[4] M. Bertoldi, L. Lorenzi: *Analytical Methods for Markov Semigroups.* Pure and applied mathematics **283**, Chapman Hall/CRC Press, 2006.

[5] V.I. Bogachev, G. Da Prato, M. Röckner: *On parabolic equations for measures*, Comm. Partial Differential equations **33** (2008), 397–418.

[6] V.I. Bogachev, N.V. Krylov, M. Röckner: *Elliptic equations for measures: regularity and global bounds of densities*, J. Math. Pures Appl. (9) **85** (2006), 743–757.

[7] V.I. Bogachev, M. Röckner, S.V. Shaposhnikov: *Global regularity and bounds for solutions of parabolic equations for probability measures*, Theory Probab. Appl. **50** (2006), 561–581.

[8] V.I. Bogachev, M. Röckner, S.V. Shaposhnikov: *Estimates of densities of stationary distributions and transition probabilities of diffusion processes*, Theory Probab. Appl. **52** (2008), 209–236.

[9] P. Cannarsa, V. Vespri: *Generation of analytic semigroups by elliptic operators with unbounded coefficients*, SIAM J. Math. Anal. **18** (1987), 857–872.

[10] P. Cannarsa, V. Vespri: *Generation of analytic semigroups in the L^p–topology by elliptic operators in $\mathbb{R}^N$*, Israel J. Math. **61** (1988), 235–255.

[11] A. Carbonaro, G. Metafune, C. Spina: *Parabolic Schrödinger operators*, J. Math. Anal. Appl. **343** (2008), 965–974.

[12] S. Cerrai: *Second order PDE's in finite and infinite dimensions. A probabilistic approach*, Lecture Notes in Math. **1762**, Springer Verlag, Berlin, 2001.

[13] C. Chicone, Y. Latushkin: *Evolution Semigroups in Dynamical Systems and Differential Equations,* Math. Surv. Monogr. **70**, Aerican Mathematical Society, Providence RI, 1999.

[14] A. Chojnowska–Michalik, B. Goldys: *Symmetric Ornstein-Uhlenbeck semigroups and their generators*, Probab. Theory Rel. Fields **124** (2002), 459–486.

[15] G. Da Prato, A. Lunardi: *Ornstein-Uhlenbeck operators with time periodic coefficients*, J. Evol. Equ. **7** (2007), 587–614.

[16] G. Da Prato, M. Röckner: *A note on evolution systems of measures for time-dependent stochastic differential equations,* in: R. C. Dalang, M. Dozzi and F. Russo (Eds.), "Seminar on Stochastic Analysis, Random Fields and Applications V," Progress in Probability **59**, Birkhäuser, Basel, 2008, pp. 115–122.

[17] G. Da Prato, V. Vespri: *Maximal L^p regularity for elliptic equations with unbounded coefficients*, Nonlinear Analysis TMA **49** (2002), 747–755.

[18] G. Di Blasio: *Linear parabolic evolution equations in L^p-spaces,* Ann. Mat. Pura Appl. (4) **138** (1984), 55–104.

[19] G. Di Blasio: *Characterization of interpolation spaces and regularity properties for holomorphic semigroups,* Semigroup Forum **38** (1989), 179–187.

[20] D. Di Giorgio, A. Lunardi, R. Schnaubelt: *Optimal regularity and Fredholm properties of abstract parabolic operators in L^p spaces on the real line,* Proc. London Math. Soc. **91** (2005), 703–737.

[21] L.C. Evans, R.F. Gariepy: *Measure theory and fine properties of functions*, Studies in Advanced Mathematics, CRC Press, Boca Raton, 1992.

[22] K.J. Engel, R. Nagel: *One-parameter Semigroups for Linear Evolution Equations*, Graduate Texts in Mathematics **194**, Springer–Verlag, New York, 2000.

[23] E.B. Dynkin: *Three classes of infinite-dimensional diffusions,* J. Funct. Anal. **86** (1989), 75–110.

[24] S. Fornaro, L. Lorenzi: *Generation results for elliptic operators with unbounded diffusion coefficients in L^p- and C_b-spaces,* Discrete Contin. Dyn. Syst. **18** (2007), 747–772.

[25] M. Geissert, A. Lunardi: *Invariant measures and maximal L^2 regularity for nonautonomous Ornstein-Uhlenbeck equations*, J. Lond. Math. Soc. (2) **77** (2008), 719–740.

[26] M. Geissert, A. Lunardi: *Asymptotic behavior in nonautonomous Ornstein-Uhlenbeck equations*, J. Lond. Math. Soc. (2) **79** (2009), 85–106.

[27] D. Gilbarg, N.S. Trudinger: *Elliptic Partial Differential Equations of Second Order,* 2nd Edition, Springer-Verlag, New York, 1983.

[28] P. Kunstmann, L. Weis: *Maximal L_p regularity for parabolic equations, Fourier multiplier theorems and H^∞ functional calculus*, in: M. Iannelli, R. Nagel and S. Piazzera (Eds.), "Functional Analytic Methods for Evolution Equations," Lecture Notes in Math. **1855**, Springer–Verlag, 2004, pp. 65–311.

[29] M. Kunze, L. Lorenzi, A. Lunardi: *Nonautonomous Kolmogorov parabolic equations with unbounded coefficients*, Trans. Amer. Math. Soc., to appear.

[30] L. Lorenzi, A. Zamboni: *Cores for parabolic operators with unbounded coefficients*, J. Differential Equation, to appear.

[31] A. Lunardi: *Analytic Semigroups and Optimal Regularity in Parabolic Problems,* Birkhäuser, Basel, 1995.

[32] A. Lunardi: *On the Ornstein–Uhlenbeck operator in L^2 spaces with respect to invariant measures,* Trans. Amer. Math. Soc. **349** (1997), 155–169.

[33] J. Maas, J.M.A.M. van Neerven: *On the domain of non-symmetric Ornstein–Uhlenbeck operators in Banach spaces,* Infin. Dimens. Anal. Quantum Probab. Relat. Top., to appear.

[34] G. Metafune, J. Prüss, A. Rhandi, R. Schnaubelt: *The domain of the Ornstein–Uhlenbeck operator on an L^p–space with invariant measure,* Ann. Scuola Norm. Sup. Pisa Cl. Sci. (5) **1** (2002), 471–485.

[35] G. Metafune, J. Prüss, A. Rhandi, R. Schnaubelt: *L^p-regularity for elliptic operators with unbounded coefficients*, Adv. Differential Equations **10** (2005), 1131–1164.

[36] G. Metafune, C. Spina: *An integration by parts formula in Sobolev spaces,* Mediterranean J. Math. **5** (2008), 359–371.

[37] J. Prüss, A. Rhandi, R. Schnaubelt: *The domain of elliptic operators on $L^p(\mathbb{R}^d)$ with unbounded drift coefficients,* Houston J. Math. **32** (2006), 563–576.

[38] F. Räbiger, A. Rhandi, R. Schnaubelt: *Perturbation and an abstract characterization of evolution semigroups,* J. Math. Anal. Appl. **198** (1996), 516–533.

[39] M. Röckner: *L^p-analysis of finite and infinite dimensional diffusion operators*, in: G. Da Prato (Ed.), "Stochastic PDE's and Kolmogorov equations in infinite dimensions (Cetraro, 1998)," Lecture Notes in Math. **1715**, Springer-Verlag, Berlin, 1999, pp. 65–116.

[40] R. Schnaubelt: *Well–posedness and asymptotic behaviour of non–autonomous linear evolution equations,* in: A. Lorenzi and B. Ruf (Eds.), "Evolution Equations, Semigroups and Functional Analysis," Birkhäuser, 2002, pp. 311–338.

[41] J. Simon: *Compact sets in the space $L^p(0,T;B)$,* Ann. Mat. Pura Appl. (4) **146** (1987), 65–96.

[42] W. Stannat: *(Nonsymmetric) Dirichlet operators on L^1: existence, uniqueness and associated Markov processes*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (4), **28** (1999), 99–140.

M. Geissert, Technische Universität Darmstadt, Fachbereich Mathematik, Schlossgartenstr. 7, 64289 Darmstadt, Germany.
E-mail address: geissert@mathematik.tu-darmstadt.de

L. Lorenzi, Dipartimento di Matematica, Università degli Studi di Parma, Viale G.P. Usberti, 53/A, I-43100 Parma, Italy.
E-mail address: luca.lorenzi@unipr.it

R. Schnaubelt, Universität Karlsruhe, Fakultät für Mathematik, 76128 Karlsruhe, Germany.
E-mail address: schnaubelt@math.uni-karlsruhe.de